4047/01 2015

1. $f(x) = x^2(1-x) = x^2 - x^3$

$f'(x) = \frac{d}{dx} f(x)$

$= 2x - 3x^2$.

For f to be increasing function, $f'(x) > 0$.

$2x - 3x^2 > 0$.

$x(2-3x) > 0$.

$\therefore 0 < x < \frac{2}{3}$.

2. (i) At $(8,3)$, $3 = \log_a 8$.

$a^3 = 8$.

$\Rightarrow a = 2$.

At $(1,b)$, $b = \log_2 1$.

$2^b = 1$.

$\Rightarrow b = 0$.

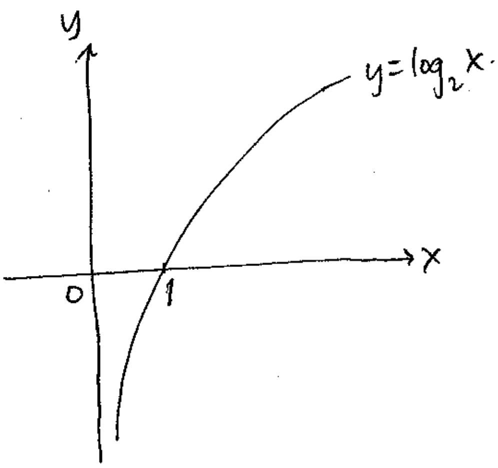
At $(c,-2)$, $-2 = \log_2 c$.

$2^{-2} = c$.

$\Rightarrow c = \frac{1}{4}$.

$\therefore a = 2, b = 0, c = \frac{1}{4}$.

2. (ii)



3. $N = N_0 e^{kt}$

When $t=3$, $N = 2N_0$.

$\Rightarrow 2N_0 = N_0 e^{3k}$

$e^{3k} = 2$.

$3k = \ln 2$

$k = \frac{\ln 2}{3}$

$\therefore k = 0.231$ (3s.f.)

4. (i) Given ax^2+6x+c is always negative, the equation

$ax^2+6x+c=0$ has no real roots.

$\Rightarrow$ Discriminant <0.

$6^2 - 4(a)(c) < 0$.

$4ac > 36$.

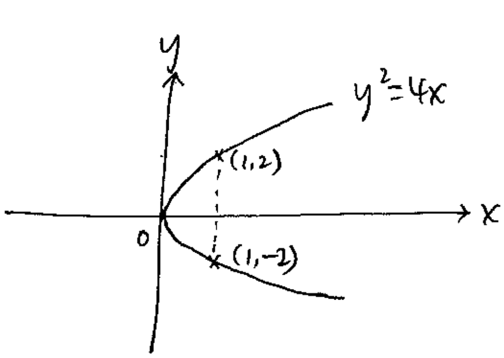
$ac > 9$. where $a<0$ for always negative condition.

$\Rightarrow c < \frac{9}{a}$.

$\therefore a<0$ and $c < \frac{9}{a}$.

4. (ii) $\therefore$ Example : $a=-3$, $c=-4$.

5. (i)



5. (ii) $y^2 = 4x$ ①

$y = x-1$ ②

Subst. ② into ① : $(x-1)^2 = 4x$.

$x^2-2x+1-4x=0$.

$x^2-6x+1=0$. ③

Solutions to ③ are x coordinates of A and B

Let x_A, x_B be x coordinates of A and B respectively.

$\Rightarrow x_A + x_B = -\frac{(-6)}{1} = 6$.

5. (ii) (continued)

At A, $y_A = x_A - 1$ where y_A is y coordinate of A.

At B, $y_B = x_B - 1$ where y_B is y coordinate of B.

$y_A + y_B = x_A - 1 + x_B - 1$

$\Rightarrow y_A + y_B = 6 - 2$
$= 4.$

Midpoint of AB is $\left(\frac{x_A+x_B}{2}, \frac{y_A+y_B}{2}\right)$, which is $\left(\frac{6}{2}, \frac{4}{2}\right)$ or $(3,2)$.

At $(3,2)$, $3+2=5$.

$\therefore$ $(3,2)$ lies on the line $x+y=5$. (shown).

6 (i) $\therefore f(x) = 4\cos^2 x - 2\sin^2 x$

$= 4\left(\frac{1+\cos 2x}{2}\right) - 2\left(\frac{1-\cos 2x}{2}\right)$

$= 2 + 2\cos 2x - 1 + \cos 2x.$

$= 3\cos 2x + 1.$

$\therefore a = 3, b = 1.$

6 (ii) $-1 \leq \cos 2x \leq 1$

$-3 \leq 3\cos 2x \leq 3$

$-2 \leq 3\cos 2x + 1 \leq 4$

$-2 \leq f(x) \leq 4$

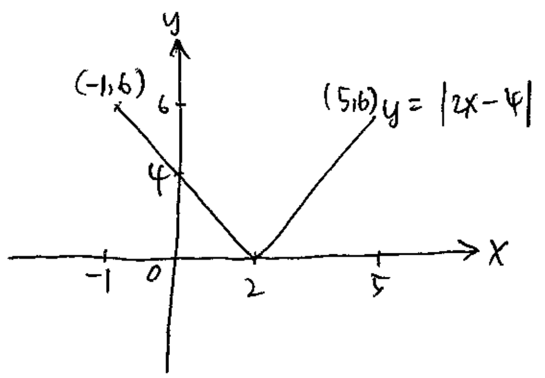
$\therefore$ Least value $= -2$, greatest value $= 4$.

6. (iii) $\therefore$ Period $= \frac{360°}{2}$ or $\frac{2\pi}{2}$

$= 180°$ or π

$\therefore$ Amplitude $= 3$.

7. (i)



7. (ii) Equation of straight line is $y = 3x - 1$.

$y = 3x - 1 \quad ①$

$y = |2x - 4| \quad ②$

Subst. ① into ②: $3x - 1 = |2x - 4|$

$2x - 4 = 3x - 1$ or $2x - 4 = 1 - 3x$

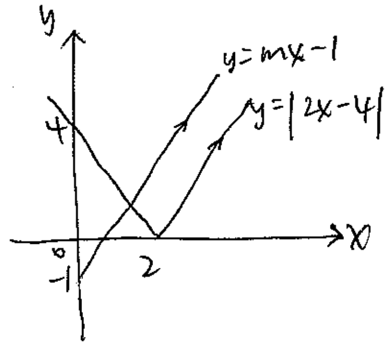
$x = -3$ or $x = 1$.

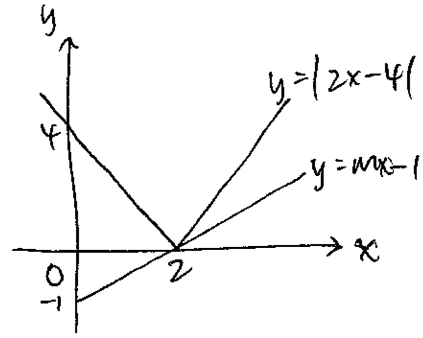
(rej $\because 3(-3) - 1 < 0$)

Subst. $x = 1$ into ①: $y = 3(1) - 1 = 2$.

$\therefore$ point of intersection is $(1, 2)$.

7. (iii)





Max value of $m = 2$ (parallel to $y = 2x - 4$)

min. value of $m = \frac{0-(-1)}{2-0}$

$= \frac{1}{2}$.

$\therefore$ Range of m is $\frac{1}{2} < m < 2$.

8. (i) $4\tan\theta + 2\cot\theta = 5\sec\theta$

$\frac{4\sin\theta}{\cos\theta} + \frac{2\cos\theta}{\sin\theta} = \frac{5}{\cos\theta}$

$\frac{4\sin\theta}{\cos\theta} - \frac{5}{\cos\theta} = \frac{-2\cos\theta}{\sin\theta}$

$\sin\theta(4\sin\theta - 5) = -2\cos\theta(\cos\theta)$

$4\sin^2\theta - 5\sin\theta = -2\cos^2\theta$

$4\sin^2\theta - 5\sin\theta = -2(1 - \sin^2\theta)$

$4\sin^2\theta - 5\sin\theta = -2 + 2\sin^2\theta$.

$2\sin^2\theta - 5\sin\theta + 2 = 0$. (shown)

8. (ii) $4\tan 2x + 2\cot 2x = 5\sec 2x$.

$2\sin^2 2x - 5\sin 2x + 2 = 0$.

Let $y = \sin 2x$.

$2y^2 - 5y + 2 = 0$.

$(2y-1)(y-2) = 0$.

$y = \frac{1}{2}$ or $y = 2$.

$\sin 2x = \frac{1}{2}$ or $\sin 2x = 2$ (rej. $\because -1 \leq \sin 2x \leq 1$).

basic $(2x) = \sin^{-1}\frac{1}{2}$

$= 30°$.

For $0° < 2x < 720°$,

$2x = 30°, 150°, 390°, 510°$.

$\therefore x = 15°, 75°, 195°, 255°$.

9. (i) $\angle DAX = \angle ACD$ (alternate segment theorem)

$\angle ACD = \angle BAC$ (alt. $\angle$s, $BA \parallel CD$)

$\angle BAC = \angle BCA$ ($BA = BC$, base $\angle$s of isosceles Δ)

$\therefore \angle DAX = \angle BCA$ (shown)

9. (ii) $\angle ABC = \angle ADX$ (ext. $\angle$ of cyclic quadrilateral)

Since $\angle DAX = \angle BCA$, by AA property, ΔABC is similar to ΔXDA.

Since $BA = BC$, $DX = DA$.

$\therefore ADX$ is isosceles.

10 (i) $v = \int a \, dt$

$= \int kt - 2 \, dt$

$= \frac{kt^2}{2} - 2t + c$ for some constant c.

When $t = 0$, $v = 30 \Rightarrow c = 30$.

When $t = 20$, $v = 10$.

$10 = \frac{k(20)^2}{2} - 2(20) + 30$

$400k = 40$.

$\therefore k = 0.1$ (shown).

10 (ii) $\therefore$ Distance between X and Y

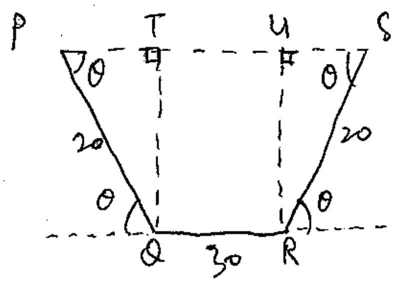
$$= \int_0^{20} \frac{0.1t^2}{2} - 2t + 30 \, dt$$

$$= \left[\frac{0.05t^3}{3} - t^2 + 30t \right]_0^{20}$$

$$= \frac{0.05(20)^3}{3} - 20^2 + 30(20) - 0.$$

$$= 333\tfrac{1}{3} \text{ m}.$$

11. (i)



Let $PT \perp TQ$ and $US \perp UR$.

$\angle TPQ = \theta = \angle USR$ (alt. $\angle$s).

$$\frac{PT}{PQ} = \cos\theta = \frac{US}{SR}.$$

$$PT = US = 20\cos\theta.$$

$$PS = PT + US + TU$$

$$= 20\cos\theta + 20\cos\theta + 30$$

$$= 30 + 40\cos\theta.$$

$$\frac{TQ}{PQ} = \sin\theta.$$

$$TQ = 20\sin\theta.$$

Area of cross section PQRS, A

$$= \frac{1}{2} \times TQ \times (PS + QR)$$

$$= \frac{1}{2}(20\sin\theta)(30 + 40\cos\theta + 30)$$

$$= 10\sin\theta(60 + 40\cos\theta).$$

$$= 600\sin\theta + 400\sin\theta\cos\theta$$

$$= 600\sin\theta + 200\sin 2\theta \text{ (shown)}.$$

11. (ii) $\frac{dA}{d\theta} = 600\cos\theta + 200(2)\cos 2\theta$.

$= 600\cos\theta + 400\cos 2\theta$.

At stationary values, $\frac{dA}{d\theta} = 0$.

$600\cos\theta + 400\cos 2\theta = 0$.

$3\cos\theta + 2\cos 2\theta = 0$.

$3\cos\theta + 2(2\cos^2\theta - 1) = 0$.

$4\cos^2\theta + 3\cos\theta - 2 = 0$.

$\cos\theta = \frac{-3 \pm \sqrt{3^2 - 4(4)(-2)}}{2(4)}$

$\cos\theta = \frac{-3+\sqrt{41}}{8}$ or $\cos\theta = \frac{-3-\sqrt{41}}{8} < -1$

(rej $-1 \leq \cos\theta \leq 1$).

basic $\theta = \cos^{-1}\left(\frac{-3+\sqrt{41}}{8}\right)$

$= 1.131$ (4 s.f.)

$\frac{d^2A}{d\theta^2} = 600(-\sin\theta) + 400(2)(-\sin 2\theta)$

$= -600\sin\theta - 800\sin 2\theta$.

When $\theta = 1.131$, $\frac{d^2A}{d\theta^2} = -1159$ (4 s.f.) < 0 (max.)

$\therefore \theta = 1.13$ (3 s.f.) for trough to hold a maximum amount of water.

12. (i) $y = \frac{36}{(2x+1)^2}$

$\frac{dy}{dx} = (-2)\frac{36}{(2x+1)^3}(2)$

$= -\frac{144}{(2x+1)^3}$

At P, $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$

$-0.36 = \frac{dy}{dx} \times 0.02$

$\frac{dy}{dx} = -18$.

$\Rightarrow -\frac{144}{(2x+1)^3} = -18$.

$(2x+1)^3 = 8$.

$2x+1 = 2 \Rightarrow x = \frac{1}{2}$.

$\therefore$ x-coordinate of P is $\frac{1}{2}$.

12. (ii) Area of region A = Area of region B.

$$\int_1^a \frac{36}{(2x+1)^2}\,dx = \int_a^4 \frac{36}{(2x+1)^2}\,dx$$

$$\left[-\frac{36}{(2)(2x+1)}\right]_1^a = \left[-\frac{36}{(2)(2x+1)}\right]_a^4$$

$$-\frac{18}{2a+1} - \left(-\frac{18}{3}\right) = -\frac{18}{9} - \left(-\frac{18}{2a+1}\right)$$

$$\frac{18}{3} + \frac{18}{9} = \frac{36}{2a+1}$$

$$2a+1 = \frac{9}{2}.$$

$$4a = 9-2.$$

$$\therefore a = \frac{7}{4} \text{ or } 1\tfrac{3}{4}.$$