4047/02 2015

1. (i) $f'(x) = 2e^{x} + e^{-2x}$.

For stationary points to exist, let $f'(x)=0$.

$2e^{x} + e^{-2x} = 0$.

$2e^{x} + \frac{1}{(e^{x})^{2}} = 0$.

Subst. $u = e^{x}$, $2u + \frac{1}{u^{2}} = 0$.

$2u = -\frac{1}{u^{2}}$

$2u^{3} = -1$.

$u^{3} = -\frac{1}{2}$.

$u = \sqrt[3]{-\frac{1}{2}}$.

$\Rightarrow e^{x} = \sqrt[3]{-\frac{1}{2}}$

Since $e^{x} > 0$, and $\sqrt[3]{-\frac{1}{2}} < 0$, there is no solution for x in $f'(x)=0$.

$\therefore$ y has no stationary points.

1. (ii) $y = \int f'(x)\,dx$

$= \int 2e^{x} + e^{-2x}\,dx$

$= 2e^{x} - \frac{1}{2}e^{-2x} + c$. for some constant c.

At $(0,2)$, $2 - \frac{1}{2} + c = 2$.

$\Rightarrow c = \frac{1}{2}$

$\therefore y = f(x)$

$= 2e^{x} - \frac{1}{2}e^{-2x} + \frac{1}{2}$.

2. (i) $\therefore \frac{d}{dx}\ln(\cos x)$

$= \frac{1}{\cos x}(-\sin x)$

$= -\tan x$ (shown)

2. (ii) $\therefore \frac{d}{dx}x\tan x$

$= \tan x + x\sec^{2}x$

2. (iii) $\int \tan x + x\sec^2 x \, dx = x\tan x + c_1$ for some constant c_1

$\int \tan x \, dx + \int x\sec^2 x \, dx = x\tan x + c_1$

$-\ln(\cos x) + c_2 + \int x\sec^2 x \, dx = x\tan x + c_1$ for some constant c_2

$\therefore \int x\sec^2 x \, dx = x\tan x + \ln(\cos x) + c$

where $c = c_1 - c_2$

$\therefore \int_0^{\frac{\pi}{4}} x\sec^2 x \, dx$

$= \left[x\tan x + \ln(\cos x)\right]_0^{\frac{\pi}{4}}$

$= \frac{\pi}{4}\tan\frac{\pi}{4} + \ln\left(\cos\frac{\pi}{4}\right) - 0 - \ln(\cos 0)$

$= \frac{\pi}{4} + \ln\left(\frac{\sqrt{2}}{2}\right)$

$= \frac{\pi}{4} + \ln\sqrt{2} - \ln 2$

$= \frac{\pi}{4} + \frac{1}{2}\ln 2 - \ln 2$

$= \frac{\pi}{4} - \frac{1}{2}\ln 2$ (shown)

---

3. (i) $y = -x^2 + 4x - 6$.

$\frac{dy}{dx} = -2x + 4$.

At $x = 1$, $\frac{dy}{dx} = 2$, $y = -3$.

Equation of tangent at P is

$y - (-3) = 2(x - 1)$

$y = 2x - 5$

At A, $y = 0$. $2x - 5 = 0 \Rightarrow x = \frac{5}{2}$.

At B, $x = 0$. $y = -5$.

A is $\left(\frac{5}{2}, 0\right)$ and B is $(0, -5)$.

$\therefore$ Area of $\triangle OAB = \frac{1}{2}\begin{vmatrix} \frac{5}{2} & 0 & 0 & \frac{5}{2} \\ 0 & 0 & -5 & 0 \end{vmatrix}$

$= \frac{1}{2}(0) - \frac{1}{2}(-5)\left(\frac{5}{2}\right)$

$= 6\frac{1}{4}$ units^2.

3. (ii) Gradient of tangent at $P = 2$.

Gradient of normal at $Q = 2$

Gradient of tangent at $Q = -\frac{1}{2}$ (tangent $\perp$ normal)

Let $\frac{dy}{dx} = -\frac{1}{2}$.

$-2x + 4 = -\frac{1}{2}$.

$x = 2\frac{1}{4}$.

$y = -(2\frac{1}{4})^2 + 4(2\frac{1}{4}) - 6$

$= -2\frac{1}{16}$.

$\therefore$ Coordinates of Q is $(2\frac{1}{4}, -2\frac{1}{16})$.

4. (a) (i) $\therefore (1+x)^9 = 1 + \binom{9}{1}(x) + \binom{9}{2}(x)^2 + \binom{9}{3}(x)^3 + \cdots$

$= 1 + 9x + 36x^2 + 84x^3 + \cdots$

4. (a) (ii) $(1 + z - z^2)^9 = 1 + 9(z - z^2) + 36(z - z^2)^2 + 84(z - z^2)^3 + \cdots$

$= 1 + 9z - 9z^2 + 36z^2 - 72z^3 + \cdots + 84(z^3 + \cdots) + \cdots$

$\therefore$ Coefficient of $z^3 = -72 + 84$

$= 12$.

4. (b) (i) $\therefore$ General term of $\left(2x + \frac{1}{3x^3}\right)^{10}$

$= \binom{10}{r}(2x)^{10-r}\left(\frac{1}{3x^3}\right)^r$

$= \binom{10}{r}(2)^{10-r}\left(\frac{1}{3}\right)^r(x)^{10-r}(x^{-3})^r$

$= \binom{10}{r}(2)^{10-r}\left(\frac{1}{3}\right)^r(x)^{10-4r}$

4(b) (ii) $\therefore$ power of x in general term $= 10 - 4r$.

4(b) (iii) Let $10 - 4r = 2$ for x^2 term

$\Rightarrow r = 2$

$\therefore$ Coefficient of x^2 in $\left(2x + \frac{1}{3x^3}\right)^{10}$

$= \binom{10}{2}(2)^{10-2}\left(\frac{1}{3}\right)^2$

$= 1280$.

5·(i)

$$\therefore \frac{11\sqrt{3}}{2\sqrt{3}+1} \times \frac{2\sqrt{3}-1}{2\sqrt{3}-1}$$

$$= \frac{22(3) - 11\sqrt{3}}{(2\sqrt{3})^2 - 1^2}$$

$$= \frac{66 - 11\sqrt{3}}{11}$$

$= 6 - \sqrt{3}$. where $a = 6$, $b = -1$.

5·(ii) $\therefore BC^2 = AC^2 - AB^2$ (Pythagoras' Theorem)

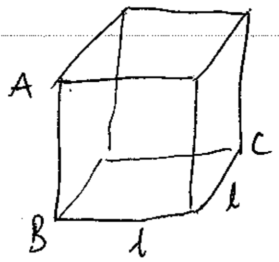
$$= \left(\frac{11\sqrt{3}}{2\sqrt{3}+1}\right)^2 - (\sqrt{3}+1)^2$$

$$= (6 - \sqrt{3})^2 - (3 + 2\sqrt{3} + 1)$$

$$= 36 - 12\sqrt{3} + 3 - 4 - 2\sqrt{3}$$

$= 35 - 14\sqrt{3}$ where $c = 35$, $d = -14$.

5·(iii)



Let l be the length of each side of square base.

$l^2 + l^2 = BC^2$ (Pythagoras' Theorem)

$2l^2 = 35 - 14\sqrt{3}$

$l^2 = \frac{35}{2} - 7\sqrt{3}$

$\therefore$ Volume of cuboid in cm^3

$$= l^2 \times AB$$

$$= \left(\frac{35}{2} - 7\sqrt{3}\right)(\sqrt{3}+1)$$

$$= \frac{35}{2}\sqrt{3} - 7\sqrt{3} + \frac{35}{2} - 7(3)$$

$$= \frac{21}{2}\sqrt{3} - \frac{7}{2}$$

$= \frac{7}{2}(3\sqrt{3} - 1)$ where $k = -1$.

6. (i) $y = \frac{2x^2}{x-1}$

$$\frac{dy}{dx} = \frac{(x-1)(4x) - (2x^2)(1)}{(x-1)^2}$$

$$= \frac{4x^2 - 4x - 2x^2}{(x-1)^2}$$

$$= \frac{2x^2 - 4x}{(x-1)^2}$$

At stationary values, $\frac{dy}{dx} = 0$.

$$\frac{2x^2 - 4x}{(x-1)^2} = 0.$$

$2x^2 - 4x = 0$.

$2x(x-2) = 0$.

$x = 0$ or $x = 2$.

$y = 0$ or $y = 8$.

$\therefore (0,0)$ and $(2,8)$ are the stationary points of y.

6. (ii) $\therefore \frac{d^2y}{dx^2} = \frac{(x-1)^2(4x-4) - 2(2x^2-4x)(x-1)(1)}{(x-1)^4}$

$$= \frac{4(x-1)^3 - 4x(x-2)(x-1)}{(x-1)^4}$$

$$= \frac{4(x-1)^2 - 4x(x-2)}{(x-1)^3}$$

$$= \frac{4(x^2 - 2x + 1 - x^2 + 2x)}{(x-1)^3}$$

$$= \frac{4}{(x-1)^3}$$

At $(0,0)$, $\frac{d^2y}{dx^2} = -4 < 0$ (max)

At $(2,8)$, $\frac{d^2y}{dx^2} = 4 > 0$ (min)

$\therefore (0,0)$ is a maximum point, $(2,8)$ is a minimum point.

7. (i) $\because$ x and y coordinates are equal and positive.

7. (ii) Let centre of C be (a,a).

Since x- and y-axes are tangents to C, radius $= a$.

$\Rightarrow (x-a)^2+(y-a)^2=a^2$ is the equation of C.

At $(9,8)$,

$(9-a)^2+(8-a)^2=a^2$.

$81-18a+a^2+64-16a+a^2=a^2$.

$a^2-34a+145=0$.

$(a-29)(a-5)=0$.

$a=29$ or $a=5$

rej. $\because (29,29)$ is above and right of $(9,8)$.

$\Rightarrow$ centre of C is $(5,5)$, radius $= 5$.

$\therefore$ Equation of C is $(x-5)^2+(y-5)^2=25$.

7. (iii) gradient of radius from $(9,8)$ to centre of C

$= \dfrac{8-5}{9-5}$

$= \dfrac{3}{4}$.

Gradient of tangent at $(9,8)$

$= -\dfrac{4}{3}$ (tan $\perp$ rad.)

$\therefore$ Equation of tangent at $(9,8)$ is

$y-8=-\frac{4}{3}(x-9)$

$y=-\frac{4}{3}x+20$.

$3y+4x=60$.

8. (i) Let $f(x)=2x^3-3x^2-5$.

When $f(x)$ is divided by $2x+1$, remainder $= f(-\frac{1}{2})$.

$f(-\frac{1}{2})=2(-\frac{1}{2})^3-3(-\frac{1}{2})^2-5$

$=-6$.

$\therefore$ Remainder $= -6$.

8 (ii) $f(-\frac{1}{2}) = -6.$

$f(-\frac{1}{2}) + 6 = 0.$

$\Rightarrow f(x)+6$ is divisible by $2x+1$.

Let $f(x)+6$

$= 2x^3 - 3x^2 - 5 + 6$

$= 2x^3 - 3x^2 + 1$

$= (2x+1)(x^2+ax+1)$ for some constant a.

Comparing coefficients of x,

$2 + a = 0.$

$a = -2.$

$\therefore 2x^3 - 3x^2 + 1 = (2x+1)(x^2-2x+1)$

$= (2x+1)(x-1)^2.$

8 (iii) Let $\frac{4-5x-8x^2}{2x^3-3x^2+1} = \frac{4-5x-8x^2}{(2x+1)(x-1)^2}$

$= \frac{A}{2x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$

for some constants, A, B and C.

$\Rightarrow 4-5x-8x^2 = A(x-1)^2 + B(2x+1)(x-1) + C(2x+1).$

Subst. $x=1$: $4-5(1)-8(1)^2 = C(2+1) \Rightarrow C = -3.$

Subst. $x=-\frac{1}{2}$: $4-5(-\frac{1}{2})-8(-\frac{1}{2})^2 = A(-\frac{1}{2}-1)^2$

$\frac{9}{4}A = \frac{9}{2} \Rightarrow A = 2.$

Subst. $x=0$: $4 = 2(-1)^2 + B(1)(-1) - 3(1)$

$-B = 5 \Rightarrow B = -5.$

$\therefore \frac{4-5x-8x^2}{2x^3-3x^2+1} = \frac{2}{2x+1} - \frac{5}{x-1} - \frac{3}{(x-1)^2}.$

9. (i) $\frac{AD}{AB} = \frac{DC}{BC} = \cos\theta$.

$AD = AB\cos\theta$
$= 80\cos\theta$
$= DC$ (D is midpoint of AC).

$\frac{BD}{AB} = \sin\theta$.

$BD = AB\sin\theta$
$= 80\sin\theta$.

$\therefore$ Length of fencing, $L = AD + DC + AB + BC + BD$
$= 80\cos\theta + 80\cos\theta + 80 + 80 + 80\sin\theta$
$= 160 + 80\sin\theta + 160\cos\theta$ (shown)

$\therefore p = 160,\ q = 80,\ r = 160$.

9. (ii) $\therefore L = 160 + 80\sin\theta + 160\cos\theta$
$= 160 + \sqrt{80^2 + 160^2}\sin\left(\theta + \tan^{-1}\frac{160}{80}\right)$
$= 160 + \sqrt{32000}\sin(\theta + \tan^{-1}2)$
$\approx 160 + 80\sqrt{5}\sin(\theta + 1.107)$

where $R = 80\sqrt{5}$ and $\alpha = 1.11$ (3 s.f.)

9. (iii) Let $L = 310$.

$160 + 80\sqrt{5}\sin(\theta + 1.107) = 310$.

$\sin(\theta + 1.107) = \frac{15}{8\sqrt{5}}$

basic $\angle(\theta + 1.107) = 0.9946$ (4 s.f.)

$\Rightarrow \theta + 1.107 = \pi - 0.9946$

$\therefore \theta = \pi - 0.9946 - 1.107$
$= 1.04$ (3 s.f.)

10. (i) $2x^2 - 6x + 5 = 0$.

$\alpha + \beta = 3$.

$\alpha\beta = \frac{5}{2}$

$\therefore \alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$
$= 3^2 - 2\left(\frac{5}{2}\right)$
$= 4$.

10. (ii) $\therefore \alpha^3+\beta^3 = (\alpha+\beta)(\alpha^2-\alpha\beta+\beta^2)$

$= (3)(4-\frac{5}{2})$

$= \frac{9}{2}$. (shown)

10. (iii) $\alpha^2+\beta+\alpha+\beta^2 = (\alpha^2+\beta^2)+(\alpha+\beta)$

$= 4+3$

$= 7.$

$(\alpha^2+\beta)(\alpha+\beta^2) = \alpha^3+\alpha^2\beta^2+\alpha\beta+\beta^3$

$= \alpha^3+\beta^3+(\alpha\beta)^2+\alpha\beta$

$= \frac{9}{2}+(\frac{5}{2})^2+\frac{5}{2}$

$= \frac{53}{4}.$

$\therefore$ Equation with roots $\alpha^2+\beta$ and $\alpha+\beta^2$ is

$x^2-7x+\frac{53}{4}=0.$

$4x^2-28x+53=0.$

11. (i) $V=x(px^2+q).$

$\Rightarrow \frac{V}{x}=px^2+q.$

Plot $\frac{V}{x}$ against x^2: gradient of straight line graph $=p$

$\frac{V}{x}$ intercept of graph $=q$.

x	5	10	15	20
V	175	650	1725	3700
$\frac{V}{x}$	35	65	115	185
x^2	25	100	225	400

x^2 axis : 1 cm to 25 units (cm^2)

$\frac{V}{x}$ axis : 1 cm to 10 units (cm^2).

From graph, gradient of straight line $= \frac{115-65}{225-100}$

$= \frac{50}{125}$

$= \frac{2}{5}.$

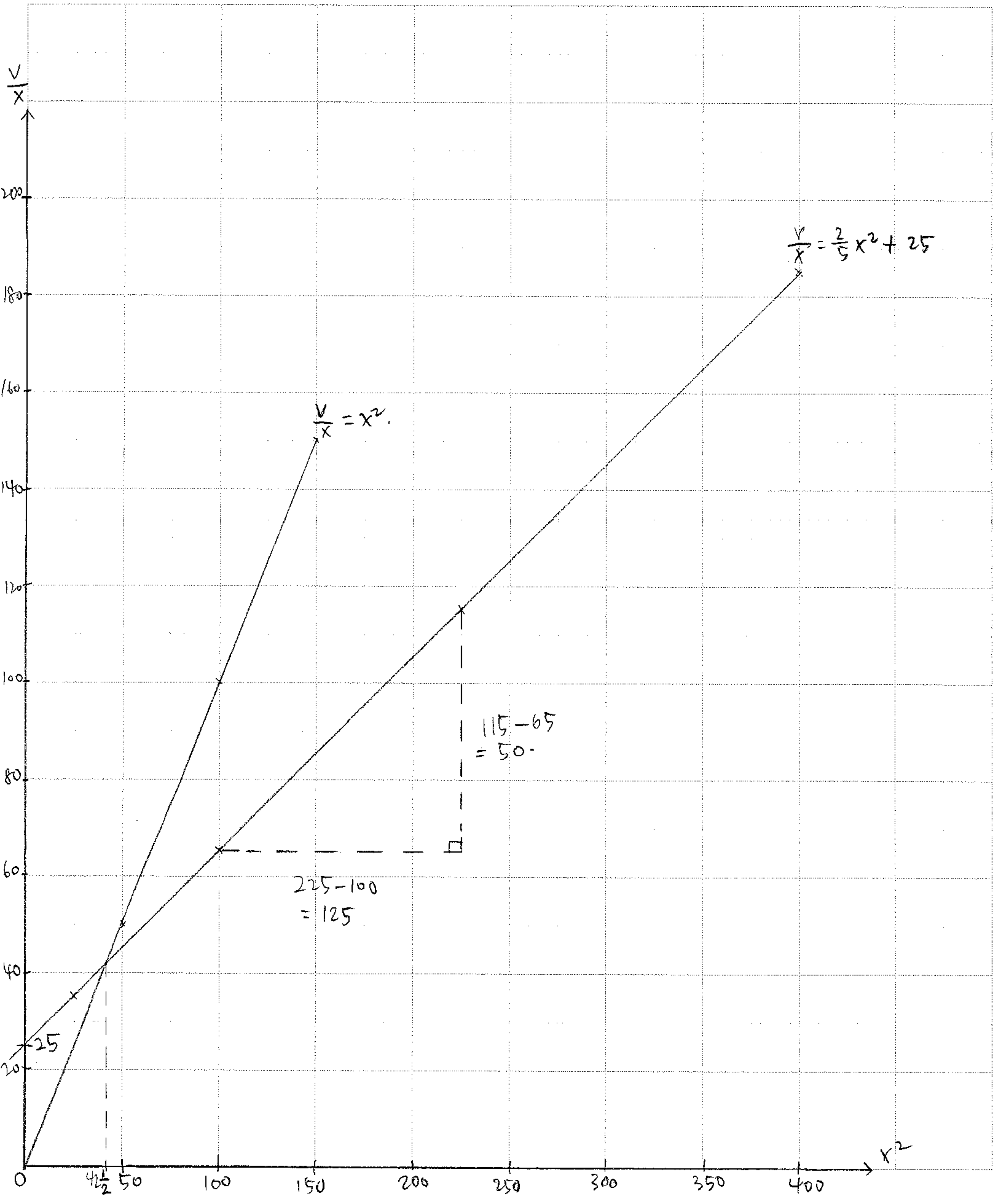
$\frac{V}{x}$ intercept $= 25.$

$\therefore p=\frac{2}{5}, q=25.$

x^2-axis 1 cm : 25 cm^2
$\frac{V}{x}$ axis : 1 cm : 10 cm^2

Name: Index No.:

Subject: Q 11. Graph of $\frac{V}{x}$ against x^2 Class: Date:



11. (ii) $V = x\left(\frac{2}{5}x^2 + 25\right)$

For V to be volume of cube, $x^2 = \frac{2}{5}x^2 + 25$.

$$\frac{3}{5}x^2 = 25.$$

$$x^2 = \frac{125}{3}.$$

Since $x > 0$, $x = \frac{5\sqrt{5}}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$

$$\therefore x = \frac{5}{3}\sqrt{15} \approx 6.5$$

11. (iii) Let $V = x^3$

$$\frac{V}{x} = x^2.$$

Gradient of $\frac{V}{x}$ against x^2 line $= 1$.

$\frac{V}{x}$ intercept $= 0$.

Plot $\frac{V}{x} = x^2$ line, the point of intersection between $\frac{V}{x} = x^2$ and $\frac{V}{x} = px^2 + q$ $(p = \frac{2}{5}, q = 25)$ indicates the coordinates of x^2 and $\frac{V}{x}$ values where cuboid is a cube.

x^2	0	50	100	150
$\frac{V}{x}$	0	50	100	150

From graph, point of intersection's x^2 coordinate $= 42\frac{1}{2}$.

$$x^2 = 42.5$$

Since $x > 0$, $x = \sqrt{42.5}$

$\therefore x = 6.5$ (1d.p.) (verified.)