2017 CCHM S4 AM Prelim P2

1. (i) $y = \frac{e^{2x}}{3x}$

$\therefore \frac{dy}{dx} = \frac{3xe^{2x}(2) - 3e^{2x}}{(3x)^2}$

$= \frac{e^{2x}(6x-3)}{9x^2}$

$= \frac{e^{2x}(2x-1)}{3x^2}$.

1. (ii) When $x=2$, $\frac{dy}{dx} = \frac{e^4(3)}{3(2)^2}$

$= \frac{1}{4}e^4$.

$\therefore \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$

$= \frac{1}{4}e^4 (0.05)$

$= \frac{1}{80}e^4$ units/sec or 0.682 units/sec (3s.f.)

1. (iii) At stationary points, $\frac{dy}{dx} = 0$.

$\frac{e^{2x}(2x-1)}{3x^2} = 0$.

Since $3x^2 \neq 0$, $e^{2x} \neq 0$, $2x-1=0$.

$\therefore$ $x = \frac{1}{2}$ is the only solution to $\frac{dy}{dx} = 0$, thus y has only one stationary point.

2. (i) $2x^2 - 3x + 5 = 0$.

$\Rightarrow \frac{1}{\alpha} + \frac{1}{\beta} = \frac{3}{2}$

$\frac{\beta+\alpha}{\alpha\beta} = \frac{3}{2}$

$\alpha+\beta = \frac{3}{2}\alpha\beta$ ①

$\Rightarrow \left(\frac{1}{\alpha}\right)\left(\frac{1}{\beta}\right) = \frac{5}{2}$

$\alpha\beta = \frac{2}{5}$ ②

2\. (i) (continued)

(2) $\rightarrow$ (1): $\alpha+\beta = \frac{3}{2}\left(\frac{2}{5}\right)$

$\therefore \alpha+\beta = \frac{3}{5}$ (shown).

2\. (ii) $\therefore \alpha^3+\beta^3$

$= (\alpha+\beta)(\alpha^2-\alpha\beta+\beta^2)$

$= (\alpha+\beta)\left[(\alpha+\beta)^2 - 2\alpha\beta - \alpha\beta\right]$

$= \frac{3}{5}\left[\left(\frac{3}{5}\right)^2 - 3\left(\frac{2}{5}\right)\right]$

$= -\frac{63}{125}$.

2\. (iii) $\alpha^2+2\beta+\beta^2+2\alpha = \alpha^2+\beta^2+2(\alpha+\beta)$

$= (\alpha+\beta)^2 - 2\alpha\beta + 2(\alpha+\beta)$

$= \left(\frac{3}{5}\right)^2 - 2\left(\frac{2}{5}\right) + 2\left(\frac{3}{5}\right)$

$= \frac{19}{25}$.

$(\alpha^2+2\beta)(\beta^2+2\alpha) = \alpha^2\beta^2 + 2\alpha^3 + 2\beta^3 + 4\alpha\beta$

$= (\alpha\beta)^2 + 2(\alpha^3+\beta^3) + 4\alpha\beta$

$= \left(\frac{2}{5}\right)^2 + 2\left(-\frac{63}{125}\right) + 4\left(\frac{2}{5}\right)$

$= \frac{94}{125}$.

$\therefore$ Equation with roots $\alpha^2+2\beta$ and $\beta^2+2\alpha$ is

$x^2 - \left(\frac{19}{25}\right)x + \frac{94}{125} = 0$. or

$125x^2 - 95x + 94 = 0$.

3. (i) General term in $\left(x - \frac{1}{2x^3}\right)^8$

$= \binom{8}{r} (x)^{8-r} \left(-\frac{1}{2x^3}\right)^r$

$= \binom{8}{r} x^{8-r} \left(-\frac{1}{2}\right)^r x^{-3r}$

$= \binom{8}{r} \left(-\frac{1}{2}\right)^r x^{8-4r}$

For constant term, $8-4r=0$. $\Rightarrow r=2$.

$\therefore$ constant term $= \binom{8}{2}\left(-\frac{1}{2}\right)^2$

$= 7$.

3. (ii) For $\frac{1}{x^4}$ term in $\left(x - \frac{1}{2x^3}\right)^8$, $8-4r=-4$. $\Rightarrow r=3$.

$\frac{1}{x^4}$ term in $\left(x-\frac{1}{2x^3}\right)^8$ is $\binom{8}{3}\left(-\frac{1}{2}\right)^3 x^{-4} = -\frac{7}{x^4}$.

General term in $\left(1+\frac{4}{x^2}\right)^n$

$= \binom{n}{r}\left(\frac{4}{x^2}\right)^r$

$= \binom{n}{r} 4^r \cdot x^{-2r}$

For $\frac{1}{x^4}$ term in $\left(1+\frac{4}{x^2}\right)^n$, $-2r=-4$ $\Rightarrow r=2$.

$\frac{1}{x^4}$ term in $\left(1+\frac{4}{x^2}\right)^n$ is $\binom{n}{2}(4)^2 x^{-4} = \frac{n(n-1)}{2} \cdot 16 \cdot x^{-4}$

$= \frac{8n(n-1)}{x^4}$

Given coefficient of $\frac{1}{x^4}$ in $\left(1+\frac{4}{x^2}\right)^n + \left(x-\frac{1}{2x^3}\right)^8$ is 1049,

$-7 + 8n(n-1) = 1049$.

$8n^2 - 8n - 1056 = 0$.

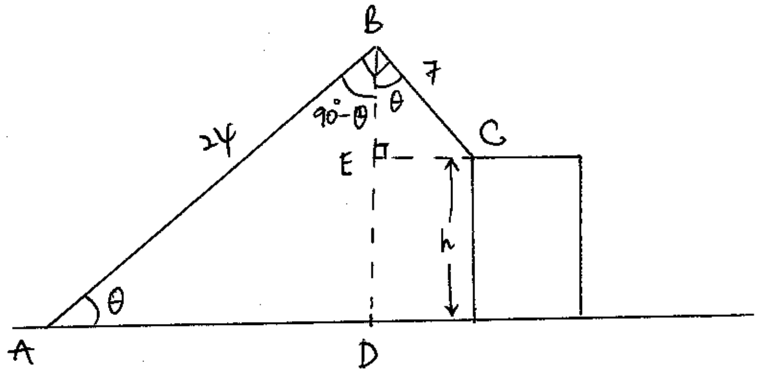
$n^2 - n - 132 = 0$.

$(n-12)(n+11) = 0$.

$n = 12$ or -11 (rej. $n>0$)

$\therefore n = 12$.

4. (i)



$$\frac{BD}{AB} = \sin\theta$$

$BD = 24\sin\theta$.

$$\frac{BE}{BC} = \cos\theta$$

$BE = 7\cos\theta$.

$\therefore h = BD - BE$

$= 24\sin\theta - 7\cos\theta$ (shown)

$\therefore p = 24,\ q = -7$.

4. (ii) $\therefore h = 24\sin\theta - 7\cos\theta$

$= \sqrt{24^2+7^2}\,\sin\left(\theta - \tan^{-1}\frac{7}{24}\right)$

$\approx 25\sin(\theta - 16.3°)$, $R = 25$, $\alpha = 16.3°$ (1 d.p.)

4. (iii) When $h = 15$, $25\sin(\theta - 16.3°) = 15$.

$\sin(\theta - 16.3°) = \frac{3}{5}$.

$\theta - 16.3° = \sin^{-1}\frac{3}{5}$

$= 36.87°$ (2 d.p.)

$\therefore \theta = 53.1°$ (1 d.p.).

5. (i) Let $v = 0$.

$t - 9 + \frac{16}{t+1} = 0$.

$(t-9)(t+1) + 16 = 0$.

$t^2 - 8t + 7 = 0$.

$(t-1)(t-7) = 0$.

$t = 1$ or 7.

$\Rightarrow t = 1$ at point A, $t = 7$ at point B.

5. (i) (continued)

$$a = \frac{dv}{dt}$$

$$= 1 - \frac{16}{(t+1)^2}$$

when $t=1$, $a = 1 - \frac{16}{(1+1)^2}$

$$= -3 .$$

$\therefore$ acceleration of particle at $A = -3 \text{ m/s}^2$.

5. (ii) displacement, $s = \int v \, dt$

$$= \int t - 9 + \frac{16}{t+1} \, dt$$

$$= \frac{1}{2}t^2 - 9t + 16\ln(t+1) + c$$ for some constant, c

At $t=0$, $s=0$.

$\frac{1}{2}(0)^2 - 9(0) + 16\ln(0+1) + c = 0 . \Rightarrow c = 0.$

$$s = \frac{1}{2}t^2 - 9t + 16\ln(t+1)$$

when $t=1$, $s = 16\ln 2 - \frac{17}{2}$.

When $t=7$, $s = 16\ln 8 - \frac{77}{2}$.

$\therefore$ Distance between A and B $= 16\ln 2 - \frac{17}{2} - \left(16\ln 8 - \frac{77}{2}\right)$

$= 30 - 16\ln 4$ m or 7.82 m (3 s.f.)

5. (iii) When $t=2$, $s = 16\ln 3 - 16$ or 1.58 (3 s.f.)

when $t=3$, $s = 16\ln 4 - \frac{45}{2}$ or -0.319 (3 s.f.)

$\therefore$ Since the particle is only at rest when $t=1$ and $t=7$ and displacement values are opposite signs at $t=2$ and $t=3$, particle is again at O between $t=2$ and $t=3$ where $s=0$ (shown).

6. (i) $f(-1) = -8$.

$3(-1)^3 + p(-1)^2 - 8(-1) - 20 = -8.$

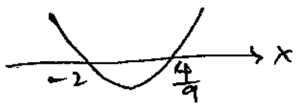
$\therefore p = 7.$

6. (ii) $f'(x) = 9x^2 + 14x - 8$.

For $f(x)$ to be decreasing, $f'(x) < 0$.

$9x^2 + 14x - 8 < 0$.

$(9x-4)(x+2) < 0$.



$\therefore$ range of x is $-2 < x < \frac{4}{9}$.

6. (iii) $f(-2) = 3(-2)^3 + 7(-2)^2 - 8(-2) - 20$.

$= 0$.

$\Rightarrow x+2$ is a factor of $f(x)$.

Let $f(x) = 3x^3 + 7x^2 - 8x - 20$

$= (x+2)(3x^2 + ax - 10)$ for some constant a.

Comparing coefficients of x, $-8 = -10 + 2a$. $\Rightarrow a = 1$.

$\therefore f(x) = (x+2)(3x^2 + x - 10)$

$= (x+2)(3x-5)(x+2)$

$= (3x-5)(x+2)^2$

6. (iv) $3(2^{3y}) + 7(2^{2y}) - 8(2^y) - 20 = 0$.

$3(2^y)^3 + 7(2^y)^2 - 8(2^y) - 20 = 0$.

$(2^y+2)^2(3 \cdot 2^y - 5) = 0$.

$2^y + 2 = 0$ (repeat) or $3 \cdot (2^y) - 5 = 0$.

Since $2^y > 0$, $2^y + 2 = 0$ has no solution.

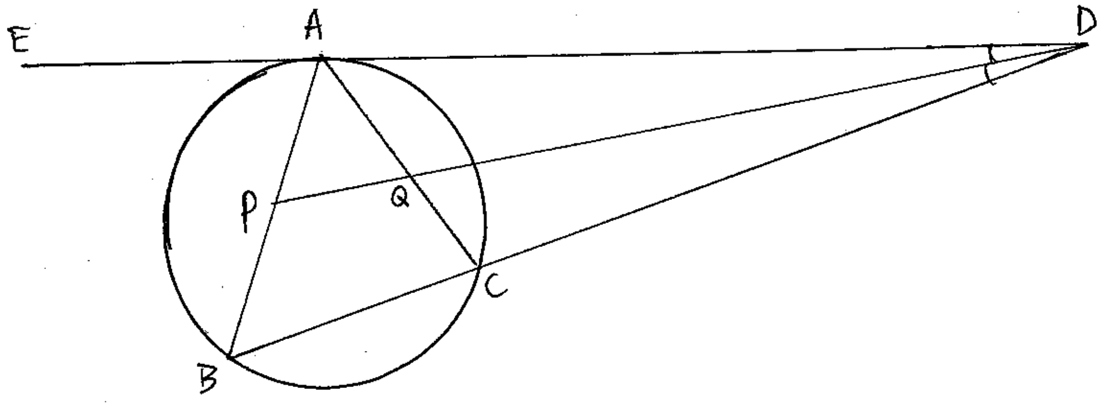
$2^y = \frac{5}{3}$.

$y \ln 2 = \ln \frac{5}{3}$.

$y = \frac{\ln \frac{5}{3}}{\ln 2}$

$\therefore y = 0.737$ (3s.f.)

7. (i)



$E\hat{A}P = A\hat{C}B$ (alternate segment theorem).

$D\hat{A}P = 180° - E\hat{A}P$ (adj. $\angle$s on a straight line)

$D\hat{C}Q = 180° - A\hat{C}B$ (adj $\angle$s on a straight line)

$\Rightarrow D\hat{A}P = D\hat{C}Q$

$A\hat{D}P = C\hat{D}Q$ (PQD bisects $A\hat{D}C$)

By AA property, $\triangle ADP$ is similar to $\triangle CDQ$.

$\Rightarrow A\hat{P}D = C\hat{Q}D$

$C\hat{Q}D = A\hat{Q}P$ (vert. opp $\angle$s).

$\Rightarrow A\hat{P}D = A\hat{P}Q = A\hat{Q}P$ (base $\angle$s of isosceles Δ)

$\Rightarrow AP = AQ$ $\therefore \triangle APQ$ is isosceles (proven).

7. (ii) Since $\triangle ADP$ is similar to $\triangle CDQ$,

$$\frac{AD}{CD} = \frac{AP}{CQ}$$

$\therefore AP \times CD = AD \times CQ$ (proven).

8. (a) $y = 1$ ①

$y = x^2 + kx + 3k - 7$ ②

① → ②: $x^2 + kx + 3k - 7 = 1$.

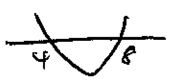
$x^2 + kx + 3k - 8 = 0$.

Since ② lies completely above ①, discriminant < 0.

$k^2 - 4(1)(3k-8) < 0$.

$k^2 - 12k + 32 < 0$.

$(k-8)(k-4) < 0$.



$\therefore$ range of k is $4 < k < 8$.

8. (b) $3\log_5 x + \log_x 5 = 2.$

$3\log_5 x + \frac{\log_5 5}{\log_5 x} = 2.$

$3\log_5 x + \frac{1}{\log_5 x} = 2.$

let $y = \log_5 x.$

$3y + \frac{1}{y} = 2.$

$\therefore 3y^2 - 2y + 1 = 0$ where $y = \log_5 x.$

Discriminant $= (-2)^2 - 4(3)(1)$

$= 4 - 12$

$= -8$

Since discriminant < 0, $3y^2 - 2y + 1 = 0$ has no real roots.

$\therefore$ There are no real solutions to $3\log_5 x + \log_x 5 = 2.$

8. (c) $\lg(x^2+8x) - \lg(x+1) = 2\lg 4.$

$\lg \frac{x^2+8x}{x+1} = \lg 16.$

$\frac{x^2+8x}{x+1} = 16.$

$x^2 + 8x - 16x - 16 = 0.$

$x^2 - 8x - 16 = 0,$

$x = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(1)(-16)}}{2}$

$= 4 + 4\sqrt{2}$. or $4 - 4\sqrt{2}$ (rej. $\because \lg(4-4\sqrt{2}+1)$ is undefined.)

$\therefore x = 4 + 4\sqrt{2}$ or 9.66 (3s.f.)

9. (i) $\therefore \frac{5\sin 2x + 2\cos 2x - 2}{1 + \cos 2x}$

$= \frac{5(2\sin x \cos x) + 2(\cos 2x - 1)}{2\cos^2 x}$

$= \frac{10\sin x \cos x + 2(-2\sin^2 x)}{2\cos^2 x}$

$= \frac{5\sin x}{\cos x} - \frac{2\sin^2 x}{\cos^2 x}$

$= 5\tan x - 2\tan^2 x$ (shown).

9. (ii) $\dfrac{5\sin 2x + 2\cos 2x - 2}{1+\cos 2x} = \dfrac{3}{4}\tan x.$

$\Rightarrow 5\tan x - 2\tan^2 x = \frac{3}{4}\tan x.$

$20\tan x - 8\tan^2 x = 3\tan x.$

$8\tan^2 x - 17\tan x = 0.$

$\tan x(8\tan x - 17) = 0.$

$\tan x = 0$ or $\tan x = \frac{17}{8}.$

$x = 0°, 180°, 360°$ basic $\angle = \tan^{-1}\frac{17}{8}$

$= 64.80°$ (2 d.p.)

$x = 64.80°, 180° + 64.80°$

$= 64.8°, 244.8°$ (1 d.p.)

$\therefore x = 0°, 64.8°, 180°, 244.8°, 360°.$

10. (i) $x^2 + y^2 + 8x + 6y = 11$

$x^2 + 8x + y^2 + 6y = 11$

$x^2 + 8x + 4^2 + y^2 + 6y + 3^2 = 4^2 + 3^2 + 11.$

$(x+4)^2 + (y+3)^2 = 36.$

$\therefore$ centre of C_1 is $(-4, -3).$

$\therefore$ radius of $C_1 = \sqrt{36}$

$= 6$ units.

10. (ii) $\therefore$ Equation of line is

$y - (-3) = 2(x - (-4))$

$y = 2x + 8 - 3$

$y = 2x + 5$ (shown).

10. (iii) Gradient of $PQ = \dfrac{15-7}{1-(-7)}$

$= 1.$

Gradient of $\perp$ bisector of chord $PQ = -1.$

midpoint of PQ is $\left(\frac{1-7}{2}, \frac{15+7}{2}\right) = (-3, 11)$

Equation of $\perp$ bisector of chord PQ is

$y - 11 = -(x - (-3))$

$y = -x + 8$ ①

$y = 2x + 5$ ②

① → ②: $-x + 8 = 2x + 5 \Rightarrow x = 1$

when $x = 1$, $y = 7.$

$\therefore$ centre of C_2 is $(1, 7).$

10. (iv) Distance between centre of C_2 and $(1,2)$.

$= \sqrt{(1-1)^2+(7-2)^2}$

$= 5$ units.

Radius of $C_2 = \sqrt{(1-1)^2+(15-7)^2}$

$= 8$ units.

Since distance between centre of C_2 and $(1,2) = 5$ units $<$ 8 units (radius of C_2), $\therefore$ $(1,2)$ lies within C_2, $a = 8$.

11. (i) $y = 2\cos\frac{x}{3}$.

$\frac{dy}{dx} = -\frac{2}{3}\sin\frac{x}{3}$

When $x = \frac{\pi}{2}$, gradient of tangent $= -\frac{2}{3}\sin\frac{\pi}{6} = -\frac{1}{3}$.

gradient of normal at $x=\frac{\pi}{2}$ is 3. (tangent $\perp$ normal)

When $x=\frac{\pi}{2}$, $y = 2\cos\frac{\pi}{6} = \sqrt{3}$.

Equation of normal at $x=\frac{\pi}{2}$ is $y - \sqrt{3} = 3\left(x - \frac{\pi}{2}\right)$.

$y = 3x - \frac{3\pi}{2} + \sqrt{3}$.

When $x = 0$, $y = -\frac{3\pi}{2} + \sqrt{3}$ (y intercept of normal at $x=\frac{\pi}{2}$)

When $y=0$, $3x = \frac{3\pi}{2} - \sqrt{3}$

$x = \frac{\pi}{2} - \frac{\sqrt{3}}{3}$. (x intercept of normal at $x=\frac{\pi}{2}$)

$\therefore$ B is $\left(\frac{\pi}{2} - \frac{\sqrt{3}}{3}, 0\right)$ and C is $\left(0, \sqrt{3} - \frac{3\pi}{2}\right)$.

11. (ii) $\therefore$ Area of shaded region

$$= \int_0^{\frac{\pi}{2}} 2\cos\frac{x}{3} - \left(3x - \frac{3\pi}{2} + \sqrt{3}\right) dx$$

$$= \left[\frac{2\sin\frac{x}{3}}{\left(\frac{1}{3}\right)} - \frac{3x^2}{2} + \frac{3\pi}{2}x - \sqrt{3}x\right]_0^{\frac{\pi}{2}}$$

$$= 6\sin\frac{\pi}{6} - \frac{3}{8}\pi^2 + \frac{3}{4}\pi^2 - \frac{\sqrt{3}}{2}\pi - 0.$$

$$= 3 + \frac{3}{8}\pi^2 - \frac{\sqrt{3}}{2}\pi \quad \text{or } 3.98 \text{ units}^2 \text{ (3 s.f.)}$$