Dunman 2017 S4 AM P1 (EOY.)

1. $f(x) = ax^2 - 4(bx-1)$

$= ax^2 - 4bx + 4.$

Given $f(x)=0$ has equal roots, discriminant $=0$.

$(-4b)^2 - 4a(4) = 0.$

$16b^2 - 16a = 0.$

$b^2 = a.$ ①

Given $x = -\frac{1}{3}$ is a root,

$a(-\frac{1}{3})^2 - 4b(-\frac{1}{3}) + 4 = 0.$

$\frac{1}{9}a + \frac{4b}{3} + 4 = 0.$ ②

Subst. ① into ②: $\frac{1}{9}b^2 + \frac{4b}{3} + 4 = 0.$

$b^2 + 12b + 36 = 0.$

$(b+6)^2 = 0.$

$b = -6$ (repeat.)

$\Rightarrow a = (-6)^2$

$= 36.$

$\therefore a = 36, b = -6.$

2. $y = \frac{x-1}{\sqrt{2x-3}}, x > \frac{3}{2}.$

$\frac{dy}{dx} = \frac{\sqrt{2x-3} - \frac{1}{2}(x-1)(2x-3)^{-\frac{1}{2}}(2)}{2x-3}$

$= \frac{2x-3-x+1}{\sqrt{(2x-3)^3}}$

$= \frac{x-2}{\sqrt{(2x-3)^3}}.$

At stationary point(s), $\frac{dy}{dx} = 0.$

2. (continued)

$$\frac{x-2}{\sqrt{(2x-3)^3}} = 0.$$

Since $\sqrt{(2x-3)^3} \neq 0$, $x-2=0$.

$$x = 2$$

$$y = \frac{2-1}{\sqrt{4-3}}$$

$$= 1.$$

x	2^-	2	2^+
$\frac{dy}{dx}$ sign	\	—	/

(minimum).

$\therefore$ Coordinates of stationary point is $(2,1)$ and it is a minimum point.

3. (i) $\int_3^5 h(x-2)^2\,dx = 13$. for some constant. h.

$$\left[\frac{h(x-2)^{2+1}}{3}\right]_3^5 = 13.$$

$$\frac{27h}{3} - \frac{h}{3} = 13.$$

$$h = \frac{3}{2}.$$

$$\therefore \int_1^3 h(x-2)^2\,dx$$

$$= \int_1^3 \frac{3}{2}(x-2)^2\,dx$$

$$= \left[\frac{3(x-2)^3}{2(3)}\right]_1^3$$

$$= \frac{1}{2}(3-2)^3 - \frac{1}{2}(1-2)^3$$

$$= 1.$$

3. (ii) $\therefore \int_3^5 \left[h(x-2)^2 + p\right] dx$

$= \int_3^5 h(x-2)^2\,dx + \int_3^5 p\,dx$ for some constant p.

$= 13 + \left[px\right]_3^5$

$= 13 + 2p.$

4. $\frac{dh}{dt} = 2t - 10$

$h = \int \frac{dh}{dt}\, dt$

$= \int 2t - 10\ dt$

$= t^2 - 10t + c$ for some constant c.

At $t=0$, $h=25$

$c = 25$.

$\Rightarrow h = t^2 - 10t + 25$ for $0 \leq t \leq 5$.

Let $h=0$,

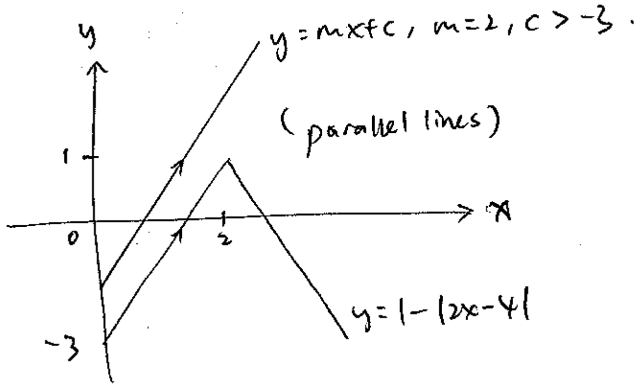
$t^2 - 10t + 25 = 0$.

$(t-5)^2 = 0$.

$t = 5$

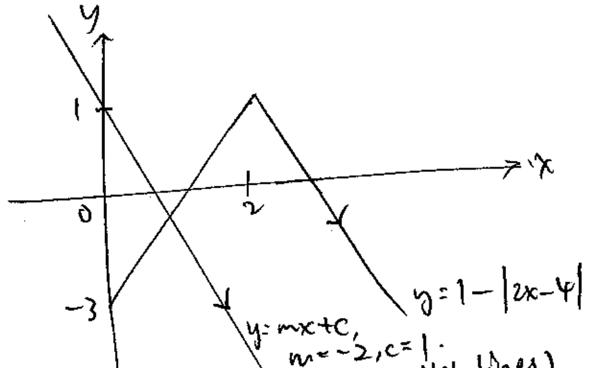
$\therefore$ Time taken for vessel to empty $= 5s$.

5.(i)



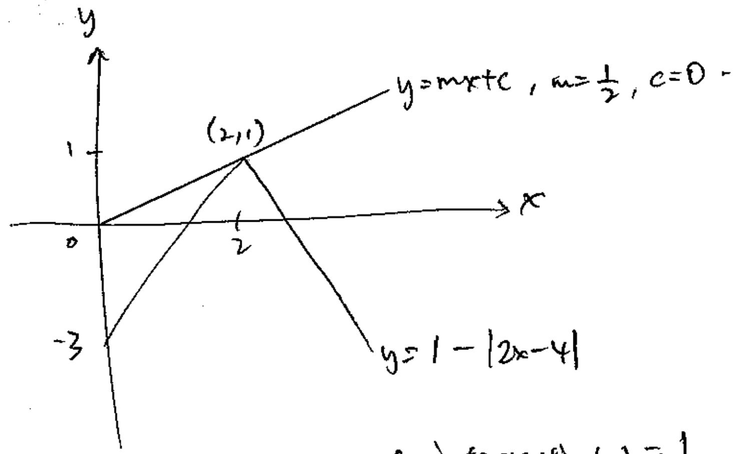
$\therefore$ From graph, number of intersection(s) = 0.

5.(ii)



$\therefore$ From graph, number of intersection(s) = 1.

5. (iii)



$\therefore$ From graph, number of intersection(s) = 1.

6. (a)

$$2^{x-2}\cdot 3^{x+2}=6^{2x}$$
$$2^x\times 2^{-2}\cdot 3^x\cdot 3^2=6^{2x}$$
$$2^x\times\frac{1}{4}\cdot 3^x\times 9=6^{2x}$$
$$\frac{9}{4}\cdot 6^x=6^{2x}$$

Since $6^x\neq 0$, $\therefore 6^x=\frac{9}{4}$.

6. (b)

$$e^{3x+1}=3^{2x-3}$$
$$\ln(e^{3x+1})=\ln(3^{2x-3})$$
$$3x+1=(2x-3)(\ln 3)$$
$$3x+1=2x\ln 3-3\ln 3$$
$$2x\ln 3-3x=3\ln 3+1.$$
$$x(2\ln 3-3)=3\ln 3+1$$
$$x=\frac{3\ln 3+1}{2\ln 3-3}$$
$$\therefore x=-5.35\ (3\text{ s.f.})$$

7. (i)

$$v=16(2e^{-2t}-e^{-t})$$
$$\therefore a=\frac{dv}{dt}$$
$$=16(-4e^{-2t}+e^{-t})$$
$$=16\left(\frac{1}{e^t}-\frac{4}{e^{2t}}\right)$$

7. (ii) When particle reaches greatest distance from O,

$v = 0$.

$16\left(\frac{2}{e^{2t}} - \frac{1}{e^t}\right) = 0$.

$\frac{2}{e^{2t}} = \frac{1}{e^t}$

$e^{2t} = 2e^t$

$e^{2t} - 2e^t = 0$.

$e^t(e^t - 2) = 0$.

Since $e^t \neq 0$, $e^t = 2$.

$t = \ln 2$.

7. (iii) $s = \int v\,dt$

$= \int 16(2e^{-2t} - e^{-t})\,dt$

$= 16\left(-\frac{1}{e^{2t}} + \frac{1}{e^t}\right) + c$. for some constant c

At $t=0$, $s=0$.

$16(-1+1) + c = 0 \Rightarrow c = 0$.

When $t = \ln 2$,

$s = 16\left(-\frac{1}{e^{2\ln 2}} + \frac{1}{e^{\ln 2}}\right)$

$= 16\left(-\frac{1}{4} + \frac{1}{2}\right)$

$= 4$

$\therefore$ Greatest distance from O = 4 m.

8. (i) Gradient of DC = gradient of AB (AB // DC)

$= \frac{11-2}{0-3}$

$= -3$.

$\therefore$ Equation of DC is

$y - 12 = -3(x - 13)$

$y = -3x + 51$.

Gradient of AD $= -\frac{1}{\text{Gradient of AB}}$ ($AB \perp AD$).

$= \frac{1}{3}$

$\therefore$ Equation of AD is

$y = \frac{1}{3}x + 11$.

8. (ii) $DC: y = -3x + 51$ ①

$AD: y = \frac{1}{3}x + 11.$ ②

① = ② : $-3x + 51 = \frac{1}{3}x + 11.$

$\Rightarrow x = 12$

$y = \frac{1}{3}(12) + 11$

$= 15.$

$\therefore D$ is $(12, 15)$.

8. (iii) $\therefore$ Area of trapezium ABCD

$$= \frac{1}{2}\begin{vmatrix} 12 & 0 & 3 & 13 & 12 \\ 15 & 11 & 2 & 12 & 15 \end{vmatrix}$$

$$= \frac{1}{2}(132 + 36 + 195) - \frac{1}{2}(33 + 26 + 144)$$

$= 80$ units^2.

9. (i) $3x^2 + 2x - 8 = 0.$

$\alpha + \beta = -\frac{2}{3}$

$\alpha\beta = -\frac{8}{3}.$

$$\therefore \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$$

$$= \left(-\frac{2}{3}\right)^2 - 2\left(-\frac{8}{3}\right)$$

$$= \frac{52}{9}.$$

9. (ii) $$\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta}$$

$$= \frac{\left(\frac{52}{9}\right)}{\left(-\frac{8}{3}\right)}$$

$$= -\frac{13}{6}$$

$\left(\frac{\alpha}{\beta}\right)\left(\frac{\beta}{\alpha}\right) = 1.$

$\therefore$ Equation with integer coefficients and roots $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$ is

$x^2 + \frac{13}{6}x + 1 = 0.$

$6x^2 + 13x + 6 = 0.$

10. (a)

$10 < x < 13.$

$7 < \frac{2x+1}{3} < 9.$

$3 \operatorname{cosec}\left(\frac{2x+1}{3}\right) = 5.$

$\sin\left(\frac{2x+1}{3}\right) = \frac{3}{5}.$

basic $\angle \left(\frac{2x+1}{3}\right) = \sin^{-1}\frac{3}{5}$

$= 0.6435$ (4 s.f.)

$\frac{2x+1}{3} = 3\pi - 0.6435$

$\therefore x = 12.7$ (3 s.f.)

10. (b)

$3 \sec\theta \tan\theta + 1 + \tan^2\theta = 0.$

$3 \sec\theta \tan\theta + \sec^2\theta = 0.$

$\sec\theta(3\tan\theta + \sec\theta) = 0.$

$\sec\theta = 0$ or $3\tan\theta + \sec\theta = 0.$

$\frac{1}{\cos\theta} = 0$ (no solution)

$\tan\theta = -\frac{\sec\theta}{3}$

$\tan\theta = -\frac{1}{3\cos\theta}$

$\sin\theta = -\frac{1}{3}$

basic $\angle(\theta) = \sin^{-1}\frac{1}{3}$

$= 19.47°$ (2 d.p.)

$\theta = 180° + 19.47°, \ 360° - 19.47°$

$\therefore \theta = 199.5°, 340.5°$ (1 d.p.)

11 (i)

$y = \frac{p}{x^2 - q}$

$x^2y - qy = p.$

$x^2y = qy + p.$

gradient of line $= q.$

x^2y - intercept $= p.$

$\therefore p = 6, \ q = 7.$

11. (ii) $xy = 1$ ①

$y = \dfrac{6}{x^2-7}$ ②

Subst. ② into ①: $\dfrac{6x}{x^2-7} = 1$

$$x^2 - 7 = 6x$$

$$x^2 - 6x - 7 = 0.$$

$$(x+1)(x-7) = 0.$$

$x = -1$ or 7
(rej. $x > 0$)

$\therefore$ positive $x = 7$.

12. (i) Volume of can $= 45\pi$.

$$\pi r^2 h + \frac{2}{3}\pi r^3 = 45\pi$$

$$r^2 h + \frac{2}{3}r^3 = 45.$$

$$r^2 h = 45 - \frac{2}{3}r^3$$

$$h = \frac{45}{r^2} - \frac{2}{3}r$$

12. (ii) $\therefore$ External surface area, A

$$= 2\pi r h + \pi r^2 + 2\pi r^2$$

$$= 2\pi r\left(\frac{45}{r^2} - \frac{2}{3}r\right) + 3\pi r^2$$

$$= \frac{90\pi}{r} - \frac{4\pi r^2}{3} + 3\pi r^2$$

$$= \frac{90\pi}{r} + \frac{5\pi r^2}{3}. \quad \text{(shown)}.$$

12 (iii) At stationary values, $\frac{dA}{dr} = 0$.

$$\frac{dA}{dr} = \frac{10}{3}\pi r - \frac{90\pi}{r^2}$$

$$\Rightarrow \frac{10}{3}\pi r - \frac{90\pi}{r^2} = 0.$$

$$10\pi r^3 = 270\pi$$

$$r^3 = 27.$$

$$r = \sqrt[3]{27} = 3.$$

$$\frac{d^2A}{dr^2} = \frac{10}{3}\pi + \frac{180\pi}{r^3}$$

When $r=3$, $\frac{d^2A}{dr^2} = \frac{10}{3}\pi + \frac{180\pi}{3^3}$

$$= 10\pi > 0 \text{ (min.)}$$

$$A = \frac{5\pi(3)^2}{3} + \frac{90\pi}{(3)}$$

$$= 45\pi$$

$\therefore$ Minimum value of $A = 45\pi$ or 141 (3s.f.)

Erratum

7 (i) $a = \frac{dv}{dt}$

$$= 16\left(\frac{1}{e^t} - \frac{4}{e^{2t}}\right)$$

When $t=2$, $a = 16\left(\frac{1}{e^2} - \frac{4}{e^4}\right)$

$$= 0.993 \text{ (3s.f.)}$$

$\therefore$ acceleration of particle at $t=2$ is $0.993\,m/s^2$.

7.(ii) $\therefore t = \ln 2$

7.(iii) $\therefore$ greatest distance from $O = 4$ m.

Erratum (continued)

11. (i) $y = \frac{p}{x^2 - q}$

$x^2y = qy + p$

gradient of line $= q$.

x^2y - intercept $= p$.

$\Rightarrow q = 5$.

Given that the line passes through $(1, 13)$,

$5(1) + p = 13$.

$p = 8$.

$\therefore p = 8, \quad q = 5$.

11. (ii) $y = \frac{8}{x^2 - 5}$. ①

$xy = 1$ ②

Subst. ① into ②: $\frac{8x}{x^2-5} = 1$.

$x^2 - 5 = 8x$

$x^2 - 8x - 5 = 0$.

For $x > 0$, $x = \frac{-(-8) + \sqrt{(-8)^2 - 4(1)(-5)}}{2}$

$= 4 + \sqrt{21}$

$\therefore x = 4 + \sqrt{21}$ or 8.58 (3 s.f.).