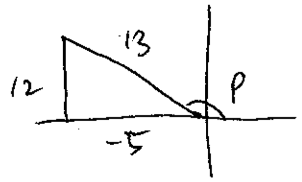
Dunman 2017 S4 AM P2 (E.O.Y.)

1. (a)



$\therefore \sin P = \frac{12}{13}$

$\therefore \cos P = -\frac{5}{13}$

1. (b) $\therefore$ LHS $= \tan 2A(2\cos A - \sec A)$

$$= \frac{\sin 2A}{\cos 2A}\left(2\cos A - \frac{1}{\cos A}\right)$$

$$= \frac{2\sin A\cos A}{\cos 2A}\left(\frac{2\cos^2 A - 1}{\cos A}\right)$$

$$= \frac{2\sin A \cancel{\cos A}}{\cancel{\cos 2A}} \cdot \frac{\cancel{\cos 2A}}{\cancel{\cos A}}$$

$= 2\sin A =$ RHS (shown).

2. (i) $y = \frac{\cos x}{2 - \sin x}$

$$\frac{dy}{dx} = \frac{(2-\sin x)(-\sin x) - (\cos x)(-\cos x)}{(2-\sin x)^2}$$

$$= \frac{-2\sin x + \sin^2 x + \cos^2 x}{(2-\sin x)^2}$$

$$= \frac{1 - 2\sin x}{(2-\sin x)^2}.$$

$\therefore a = 1, b = -2.$

2. (ii) $\therefore \int_0^{\frac{\pi}{2}} \frac{1 - 2\sin x}{(6 - 3\sin x)^2}\,dx$

$$= \int_0^{\frac{\pi}{2}} \frac{1-2\sin x}{[3(2-\sin x)]^2}\,dx$$

$$= \frac{1}{9}\int_0^{\frac{\pi}{2}} \frac{1-2\sin x}{(2-\sin x)^2}\,dx$$

$$= \frac{1}{9}\left[\frac{\cos x}{2-\sin x}\right]_0^{\frac{\pi}{2}} = -\frac{1}{18}.$$

3. (a) (i) $\therefore (1-2x)^5$

$= 1 + \binom{5}{1}(-2x) + \binom{5}{2}(-2x)^2 + \cdots$

$= 1 - 10x + 40x^2 + \cdots$

3. (a) (ii) $(a+bx)(1-2x)^5 = 2 + cx + 10x^2 + \cdots$

$\therefore a = 2.$

$\Rightarrow (2+bx)(1-10x+40x^2+\cdots) = 2 + cx + 10x^2 + \cdots$

Comparing x^2 terms,

$2(40x^2) + (bx)(-10x) = 10x^2$

$80x^2 - 10bx^2 = 10x^2.$

$\Rightarrow -10b = -70.$

$b = 7.$

Comparing x terms,

$2(-10x) + 7x(1) = cx.$

$-20 + 7 = c$

$c = -13.$

$\therefore b = 7, c = -13.$

3. (b) General term of $\left(px + \frac{q}{x}\right)^n$

$= \binom{n}{r}(px)^{n-r}\left(\frac{q}{x}\right)^r$

$= \binom{n}{r} p^{n-r} x^{n-r} q^r x^{-r}$

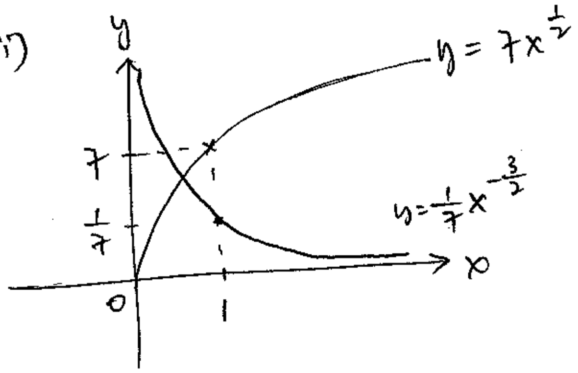
$= \binom{n}{r} p^{n-r} q^r x^{n-2r}.$

Since 4th term is independent of x, $r = 3$, $n - 2r = 0.$

$n - 2(3) = 0.$

$\therefore n = 6.$

4. (i)/(ii)



4. (iii) $7x^{\frac{1}{2}} = \frac{1}{7}x^{-\frac{3}{2}}$

$x^{\frac{1}{2}+\frac{3}{2}} = \frac{1}{49}.$

$x^2 = \frac{1}{49}$

Since $x > 0$, $x = \sqrt{\frac{1}{49}}$

$= \frac{1}{7}.$

$y = 7\left(\frac{1}{7}\right)^{\frac{1}{2}}$

$= \sqrt{7}.$

$\therefore$ point of intersection is $\left(\frac{1}{7}, \sqrt{7}\right)$.

4. (iv) For $y = 7x^{\frac{1}{2}}$, $\frac{dy}{dx} = \frac{7}{2}x^{-\frac{1}{2}}$

when $x = \frac{1}{7}$, gradient of tangent $= \frac{7}{2}\left(\frac{1}{7}\right)^{-\frac{1}{2}}$

$= \frac{7}{2}\sqrt{7}.$

For $y = \frac{1}{7}x^{-\frac{3}{2}}$, $\frac{dy}{dx} = -\frac{3}{14}x^{-\frac{5}{2}}.$

when $x = \frac{1}{7}$, gradient of tangent $= -\frac{3}{14}\left(\frac{1}{7}\right)^{-\frac{5}{2}}$

$= -\frac{21}{2}\sqrt{7}.$

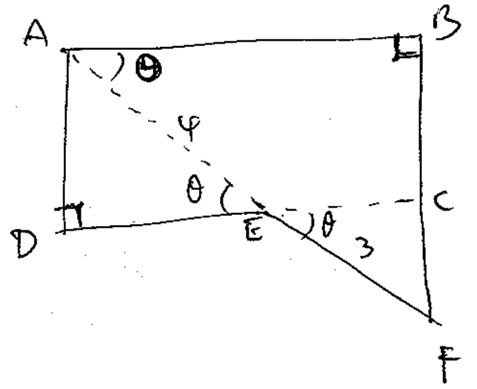
product of gradient of both tangents

$= \frac{7}{2}\sqrt{7} \times -\frac{21}{2}\sqrt{7}$

$= -\frac{1029}{4} \neq -1.$

$\therefore$ Tangents to the graphs at $\left(\frac{1}{7}, \sqrt{7}\right)$ are not perpendicular.

5.(i)



$$\frac{CF}{EF} = \sin\theta$$

$CF = EF\sin\theta = 3\sin\theta.$

$$\frac{AD}{AE} = \sin\theta$$

$AD = AE\sin\theta = 4\sin\theta$

$BC = AD = 4\sin\theta$ (rectangle property)

$$\frac{DE}{AE} = \cos\theta$$

$DE = AE\cos\theta = 4\cos\theta.$

$$\frac{EC}{EF} = \cos\theta$$

$EC = EF\cos\theta = 3\cos\theta$

$DC = DE + EC$

$= 7\cos\theta$

$= AB$ (rectangle property)

$\therefore L = FB + BA + AD + DE + EF$

$= CF + BC + 7\cos\theta + 4\sin\theta + 4\cos\theta + 3$

$= 3\sin\theta + 4\sin\theta + 11\cos\theta + 4\sin\theta + 3$

$= 11\sin\theta + 11\cos\theta + 3.$

$\therefore a = 11,\ b = 11,\ c = 3.$

5.(ii) $\therefore L = \sqrt{11^2+11^2}\,\sin\left(\theta + \tan^{-1}\frac{11}{11}\right) + 3$

$= 11\sqrt{2}\,\sin(\theta + 45°) + 3.$, $R = 11\sqrt{2}$, $\alpha = 45°.$

5.(iii) $L = 18$.

$11\sqrt{2}\sin(\theta + 45^\circ) + 3 = 18$.

$\sin(\theta+45^\circ) = \frac{15}{11\sqrt{2}}$.

basic $\angle (\theta + 45^\circ) = \sin^{-1}\frac{15}{11\sqrt{2}}$

$= 74.63^\circ$ (2 d.p.)

Since $90^\circ \le \theta + 45^\circ < 135^\circ$,

$\theta + 45^\circ = 180^\circ - 74.63^\circ$.

$\therefore \theta = 60.4^\circ$ (1 d.p.)

6. (i) Subst. $y = \frac{1}{\sqrt{2}}$ into $y = \frac{2\sqrt{2}}{\pi}x$,

$\pi = 4x$.

$x = \frac{\pi}{4}$.

$\therefore$ A is $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$.

Subst. $y = 0$ into $y = \cos x$,

$x = \frac{\pi}{2}$.

$\therefore$ B is $\left(\frac{\pi}{2}, 0\right)$.

6. (ii) Area bounded by $y = -\tan^2\frac{1}{2}x$, x-axis and $x = \frac{\pi}{2}$

$= \left| \int_0^{\frac{\pi}{2}} -\tan^2\frac{1}{2}x \, dx \right|$

$= \left| \int_0^{\frac{\pi}{2}} 1 - \sec^2\frac{1}{2}x \, dx \right|$

$= \left| \left[x - \frac{\tan\frac{1}{2}x}{\left(\frac{1}{2}\right)} \right]_0^{\frac{\pi}{2}} \right|$

$= \left| \frac{\pi}{2} - 2 \right|$

$= 2 - \frac{\pi}{2}$ units^2.

6. (ii) (continued)

Area bounded by $y = \frac{2\sqrt{2}}{\pi}x$, x-axis and $y = \cos x$

$= \frac{1}{2}\left(\frac{1}{\sqrt{2}}\right)\left(\frac{\pi}{4}\right) + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos x \, dx$

$= \frac{\sqrt{2}\pi}{16} + \left[\sin x\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$

$= \frac{\sqrt{2}}{16}\pi + 1 - \frac{\sqrt{2}}{2}$ units^2.

$\therefore$ Area of the shaded region

$= 2 - \frac{\pi}{2} + \frac{\sqrt{2}}{16}\pi + 1 - \frac{\sqrt{2}}{2}$

$= 3 + \frac{\sqrt{2}}{16}\pi - \frac{\pi}{2} - \frac{\sqrt{2}}{2}$ or 1.00 unit^2 (3 s.f.)

7. (i) $\frac{ax^3 + x^2 + 6x + b}{2x^2 + x - 1} = 3x - 1 + \frac{cx + 4}{2x^2 + x - 1}$

$\frac{ax^3 + x^2 + 6x + b}{(2x-1)(x+1)} = 3x - 1 + \frac{cx + 4}{(2x-1)(x+1)}$

$ax^3 + x^2 + 6x + b = (3x-1)(2x-1)(x+1) + cx + 4.$

Comparing x^3 terms,

$ax^3 = (3x)(2x)(x)$

$ax^3 = 6x^3$

$a = 6$

Comparing constants,

$b = (-1)(-1)(1) + 4$

$b = 5.$

Subst. $x = 1$, $6 + 1 + 6 + 5 = (2)(1)(2) + c + 4$

$c = 10.$

$\therefore a = 6, \ b = 5, \ c = 10.$

7. (ii) $$\frac{6x^3+x^2+6x+5}{2x^2+x-1} = 3x-1+\frac{10x+4}{2x^2+x-1}$$

let $$\frac{10x+4}{2x^2+x-1} = \frac{10x+4}{(2x-1)(x+1)}$$
$$= \frac{P}{2x-1}+\frac{Q}{x+1}$$ for some constants P, Q.

$10x+4 = P(x+1)+Q(2x-1)$

Subst. $x=-1$, $-10+4 = Q(-3)$

$Q = 2$.

Subst. $x=\frac{1}{2}$, $10(\frac{1}{2})+4 = P(\frac{3}{2})$

$P = 6$.

$$\therefore \int \frac{6x^3+x^2+6x+5}{2x^2+x-1}\,dx$$
$$= \int 3x-1+\frac{6}{2x-1}+\frac{2}{x+1}\,dx$$
$$= \frac{3}{2}x^2 - x + \frac{6\ln(2x-1)}{2} + \frac{2\ln(x+1)}{1} + d$$ for some constant, d.
$$= \frac{3}{2}x^2 - x + 3\ln(2x-1) + 2\ln(x+1) + d.$$

8. (i) $\therefore a = 3$, $b = \frac{\pi}{6}$, $c = 0$.

8. (ii) $\therefore$ maximum height above sea level $= 3$ m.

8. (iii) When $t = 16$,

$h = 3\sin(\frac{\pi}{6}\times 16)$

$= \frac{3\sqrt{3}}{2}$

$\therefore$ Height of tide above sea level $= \frac{3\sqrt{3}}{2}$ m.

8. (iv) $h = 2$

$3\sin(\frac{\pi}{6}t) = 2$.

$\sin(\frac{\pi}{6}t) = \frac{2}{3}$

basic $\angle$ $(\frac{\pi}{6}t) = \sin^{-1}\frac{2}{3} = 0.7297$ (4 s.f.)

$\frac{\pi}{6}t = 0.7297,\ \pi - 0.7297,\ 2\pi + 0.7297,\ 3\pi - 0.7297$

t (in hours) $= 1.394,\ 4.606,\ 13.39,\ 16.61$ (4 s.f.)

$\therefore$ Timings to cross are from 1.24 am to 4.36 am and from 1.24 pm to 4.36 pm.

(nearest min.) (nearest min.)

9. (i) $f(x) = \dfrac{x+5}{3x+1}, \ x > 0.$

$$\therefore f'(x) = \frac{d}{dx} f(x)$$
$$= \frac{d}{dx}\left(\frac{x+5}{3x+1}\right)$$
$$= \frac{(3x+1) - 3(x+5)}{(3x+1)^2}$$
$$= -\frac{14}{(3x+1)^2}$$

9. (ii) Let $f(x) = 0$.

$$\frac{x+5}{3x+1} = 0.$$

Since $3x+1 \neq 0$, $x+5=0$.
$$x = -5.$$

When $x=-5$, $f'(-5) = -\dfrac{14}{(-14)^2}$
$$= -\frac{1}{14}$$

$\Rightarrow$ gradient of normal at $x=-5$ is 14. (normal $\perp$ tangent)

$\therefore$ Equation of normal at $x=-5$ is
$$y - 0 = 14(x+5).$$
$$y = 14x + 70.$$

9. (iii) $f'(x) = -\dfrac{14}{(3x+1)^2}$ for $x > 0$.

Since $(3x+1)^2 > 0$,
$$-\frac{14}{(3x+1)^2} < 0.$$
$\Rightarrow f'(x) < 0$

$\therefore f$ is a decreasing function for $x > 0$.

9. (iv) $f''(x) = -\dfrac{14(-2)}{(3x+1)^3}(3)$
$$= \frac{84}{(3x+1)^3}.$$

For $x>0$, $(3x+1)^3 > 0$ and $\dfrac{84}{(3x+1)^3} > 0$.

Since $f''(x) > 0$,

$\therefore$ gradient of the curve is an increasing function.

10.(a) (i) $x-3y+1=0$ ①

$(x-2)^2+(y-1)^2=10$ ②

① : $x=3y-1$ ③

subst. ③ into ②:

$(3y-1-2)^2+(y-1)^2=10.$

$9y^2-18y+9+y^2-2y+1-10=0.$

$10y^2-20y=0.$

$y^2-2y=0.$

$y(y-2)=0.$

$y=0$ or $y=2$

$\Rightarrow x=-1$ or $x=5.$

$\therefore$ A and B are $(-1,0)$ and $(5,2)$.

10. (a) (ii) $(x-2)^2+(y-1)^2=10.$

Centre of circle is $(2,1)$.

Midpoint of AB is $\left(\frac{-1+5}{2}, \frac{0+2}{2}\right)$ or $(2,1)$.

Since midpoint of AB is the centre of the circle,

$\therefore$ AB is a diameter of the circle.

10. (b) $y=3x+8$ ①

$x^2+y^2-4x-8y+10=0$ ②

①→②: $(3x+8)^2+x^2-4x-8(3x+8)+10=0.$

$9x^2+48x+64+x^2-4x-24x-64+10=0.$

$10x^2+20x+10=0.$

$x^2+2x+1=0.$

Discriminant $=(2)^2-4(1)(1)$

$=0.$ (real, equal roots).

$\therefore$ $y=3x+8$ is a tangent to circle $x^2+y^2-4x-8y+10=0$ (proven).

Solving $x^2+2x+1=0$,

$(x+1)^2=0.$

$x=-1$ (repeat.)

Subst. $x=-1$ into $y=3x+8$, $y=5$

$\therefore$ point at which $y=3x+8$ is a tangent to the circle is $(-1,5)$.

11. (a) $\log_x 81 = \log_3 x$.

$\log_x 3^4 = \frac{1}{\log_x 3}$

$4(\log_x 3) = \frac{1}{\log_x 3}$

Let $\log_x 3 = y$.

$4y = \frac{1}{y}$.

$y^2 = \frac{1}{4}$

$y = \frac{1}{2}$ or $-\frac{1}{2}$.

$\log_x 3 = \frac{1}{2}$ or $\log_x 3 = -\frac{1}{2}$.

$x^{\frac{1}{2}} = 3$ or $x^{-\frac{1}{2}} = 3$.

$x = 9$. $\frac{1}{x} = 9$.

$x = \frac{1}{9}$.

$\therefore x = \frac{1}{9}$ or 9.

11. (b) (i) $T = Ta + (T_0 - Ta)e^{-kx}$.

Subst. $x = 2$, $Ta = 350$, $T_0 = 40$, $T = 125$,

$125 = 350 + (40 - 350)e^{-2k}$

$e^{-2k} = \frac{45}{62}$

$e^{2k} = \frac{62}{45}$

$2k = \ln\left(\frac{62}{45}\right)$

$k = \frac{1}{2}\ln\left(\frac{62}{45}\right)$ or 0.1602 (4 s.f.)

$\therefore k = 0.160$ (3 s.f.)

$\therefore T = Ta + (T_0 - Ta)e^{-kx}$

$= 350 + (40 - 350)e^{-0.1602x}$

$= 350 - 310e^{-0.1602x}$ (shown)

11. (b) (ii) $T = 165$.

$$350 - 310e^{-0.1602x} = 165$$

$$e^{-0.1602x} = \frac{37}{62}$$

$$-0.1602x = \ln\left(\frac{37}{62}\right)$$

$$x = -\frac{1}{0.1602}\ln\left(\frac{37}{62}\right)$$

$$= 3.222 \text{ hours} \quad (4 \text{ s.f.})$$

$$= 3 \text{ hours } 13 \text{ mins. (nearest min.)}$$

$\therefore$ Time taken = 3 hours 13 mins.

11. (c) $\log_x(p^2q^3) = m$ ①

$\log_x(pq^2) = n$ ②

①: $\log_x p^2 + \log_x q^3 = m$

$2\log_x p + 3\log_x q = m$.

$$\log_x p = \frac{m - 3\log_x q}{2} \quad ③$$

②: $\log_x p + \log_x q^2 = n$.

$\log_x p + 2\log_x q = n$. ④

Subst. ③ into ④: $\dfrac{m - 3\log_x q}{2} + 2\log_x q = n$.

$$m - 3\log_x q + 4\log_x q = 2n.$$

$$\log_x q = 2n - m. \quad ⑤$$

Subst. ⑤ into ③: $\log_x p = \dfrac{m - 3(2n-m)}{2}$

$$\log_x p = \frac{4m - 6n}{2}$$

$$= 2m - 3n$$

$\therefore \log_x \sqrt{pq}$

$= \frac{1}{2}\log_x(pq)$

$= \frac{1}{2}\log_x p + \frac{1}{2}\log_x q$

$= \frac{1}{2}(2m-3n) + \frac{1}{2}(2n-m)$

$= \dfrac{m-n}{2}$.