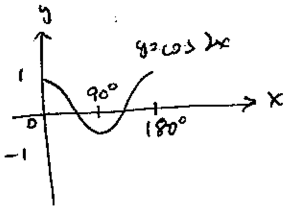
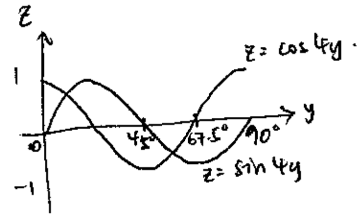
Montfort Prelim 4 AM 2017 P1

1. (i) $45° < x \leq 90°$.



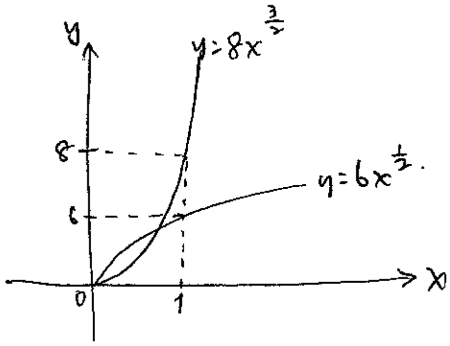
1. (ii) $67.5° < y < 90°$



1. (iii) (a) $\tan(x+y) < 0$ since
$112.5° \leq x+y \leq 180°$ (2nd quad.)

1. (iii) (b) $\tan 3y > 0$ since
$202.5° \leq 3y \leq 270°$ (3rd quad.)

2. (i)



2. (ii)
$y = 8x^{\frac{3}{2}}$ ①
$y = 6x^{\frac{1}{2}}$ ②

① → ②: $8x^{\frac{3}{2}} = 6x^{\frac{1}{2}}$

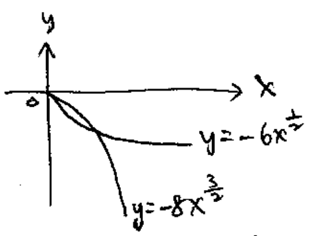
$x^{\frac{3}{2}-\frac{1}{2}} = \frac{6}{8} = \frac{3}{4}$

Subst. $x = \frac{3}{4}$ into ①: $y = 8\left(\frac{3}{4}\right)^{\frac{3}{2}}$

$= 3\sqrt{3}$.

∴ coordinates of points of intersection are $\left(\frac{3}{4}, 3\sqrt{3}\right)$ and $(0,0)$.

2. (iii)



∴ coordinates of points of intersection are $\left(\frac{3}{4}, -3\sqrt{3}\right)$ and $(0,0)$.

3. (i) When $x = \frac{1}{2}$, $|kx-1| = 0$, $y = 5$.

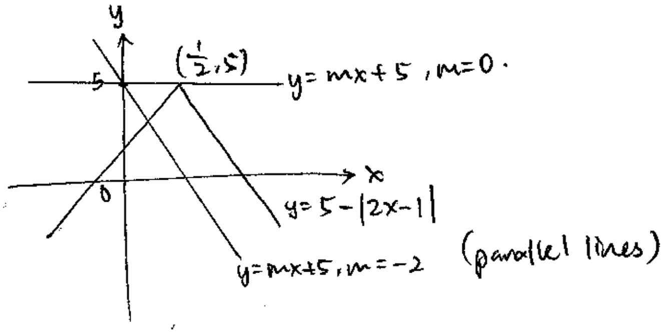
$\Rightarrow k(\frac{1}{2}) - 1 = 0$.

$\therefore k = 2$.

$\Rightarrow h - |2(\frac{1}{2}) - 1| = 5$.

$\therefore h = 5$

3. (ii) (a)



$\therefore$ set of m is $m \leq -2$ or $m = 0$.

3. (ii) (b) $\therefore$ set of m is $-2 < m < 0$.

4. (i) $\therefore$ LHS $= \dfrac{\sin^3 x - \cos^3 x}{\sin x - \cos x}$

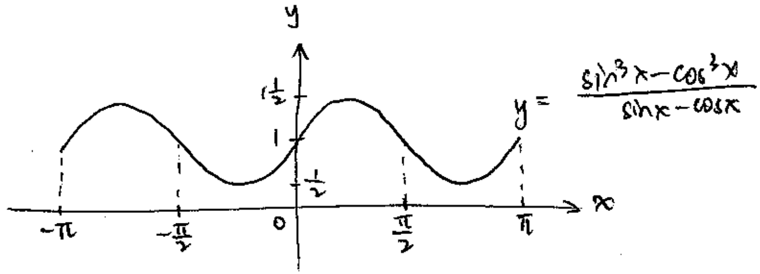
$= \dfrac{(\sin x - \cos x)(\sin^2 x + \sin x \cos x + \cos^2 x)}{\sin x - \cos x}$

$= \sin^2 x + \sin x \cos x + \cos^2 x$

$= 1 + \frac{1}{2}\sin 2x$

$=$ RHS (proven).

4. (ii)



5. (i) $\therefore \frac{d}{dx}\left[\frac{1}{4}x^2(2\ln x - 1)\right]$

$= \frac{1}{2}x(2\ln x - 1) + \frac{1}{4}x^2\left(\frac{2}{x}\right)$

$= x\ln x - \frac{1}{2}x + \frac{1}{2}x$

$= x\ln x$. (shown).

5. (ii) $\therefore \int_1^4 \frac{x\ln x}{6}\,dx$

$= \frac{1}{6}\left[\frac{1}{4}x^2(2\ln x - 1)\right]_1^4$

$= \frac{1}{6}\left(8\ln 4 - 4 - \frac{1}{2}\ln 1 + \frac{1}{4}\right)$

$= \frac{4}{3}\ln 4 - \frac{5}{8}$. or 1.22 (3s.f.)

6. (i) $-1 \leq \cos\left(\frac{2\pi t}{3}\right) \leq 1$.

$-5 \leq -5\cos\left(\frac{2\pi t}{3}\right) \leq 5$.

$5 \leq -5\cos\left(\frac{2\pi t}{3}\right) + 10 \leq 15$.

$\therefore$ maximum distance $= 15$ cm.

6. (ii) Let $y = 10$.

$\cos\left(\frac{2\pi t}{3}\right) = 0$.

$\Rightarrow \frac{2\pi t}{3} = \cos^{-1} 0 = \frac{\pi}{2}$.

$\frac{2t}{3} = \frac{1}{2}$.

$t = \frac{3}{4}$

$\therefore$ Time taken $= 0.75$ second.

6. (iii) The suspended spring eventually comes to a stopping point after some time in real life. The modelled equation $y = -5\cos\left(\frac{2\pi t}{3}\right) + 10$ is a periodic curve that continues perpetually.

7. (i) $\therefore$ Area of ΔABC in cm^2

$$= \frac{1}{2} \cdot AB \cdot BC \cdot \sin A\hat{B}C$$

$$= \frac{1}{2}(2\sqrt{6}+\sqrt{3})(7\sqrt{2}-3)\sin 120^\circ$$

$$= \frac{1}{2}\left(14\sqrt{12} - 6\sqrt{6} + 7\sqrt{6} - 3\sqrt{3}\right)\left(\frac{\sqrt{3}}{2}\right)$$

$$= \frac{\sqrt{3}}{4}\left(28\sqrt{3} - 3\sqrt{3} + \sqrt{6}\right)$$

$$= \frac{\sqrt{3}(25\sqrt{3}+\sqrt{6})}{4}$$

$$= \frac{75+3\sqrt{2}}{4}. \quad \text{(shown)}.$$

7. (ii) Let the perpendicular distance from A to BC be d cm.

$$\frac{1}{2} \cdot d \cdot BC = \frac{75+3\sqrt{2}}{4}$$

$$d = \frac{150+6\sqrt{2}}{4(7\sqrt{2}-3)}$$

$$= \frac{75+3\sqrt{2}}{14\sqrt{2}-6} \times \frac{14\sqrt{2}+6}{14\sqrt{2}+6}$$

$$= \frac{1068\sqrt{2}+534}{356}$$

$$= 3\sqrt{2} + \frac{3}{2}.$$

$\therefore$ Perpendicular distance from A to BC $= \frac{3}{2} + 3\sqrt{2}$, $a = \frac{3}{2}$, $b = 3$.

8. (a) $y = 3x + m$ ①

$y = \frac{3}{1-x}$ ②

①→②: $3x + m = \frac{3}{1-x}$

$3x + m - 3x^2 - mx = 3.$

$3x^2 + (m-3)x + 3 - m = 0.$

Given that ① is a tangent to ②, discriminant $= 0$.

$(m-3)^2 - 4(3)(3-m) = 0.$

$m^2 - 6m + 9 - 36 + 12m = 0.$

$m^2 + 6m - 27 = 0.$

$(m+9)(m-3) = 0.$

$\therefore m = -9$ or 3.

8. (b) $px^2 + (2p+q)x + 2q = 0$.

Discriminant $= (2p+q)^2 - 4p(2q)$

$= 4p^2 + 4pq + q^2 - 8pq$

$= 4p^2 - 4pq + q^2$

$= (2p-q)^2$.

Since $(2p-q)^2 \geq 0$, discriminant ≥ 0.

$\therefore$ roots of $px^2 + (2p+q)x + 2q = 0$ are real for all p and q. (shown).

$\therefore$ For real and distinct roots, $(2p-q)^2 > 0$,

all p and q are real and $p \neq \frac{q}{2}$.

9. (i) $(x+3)^2 = x^2 + 6x + 9$,

$$\begin{array}{r|l} & 3 \\ \hline x^2+6x+9 & 3x^2 + 23x - 45 \\ & -(3x^2 + 18x + 27) \\ \hline & 5x - 72 \end{array}$$

$$\frac{3x^2+23x-45}{(x+3)^2} = 3 + \frac{5x-72}{(x+3)^2}$$

Let $\frac{5x-72}{(x+3)^2} = \frac{A}{x+3} + \frac{B}{(x+3)^2}$ for some constants A and B.

$5x - 72 = A(x+3) + B$.

Comparing terms in x, $A = 5$.

Comparing constants, $-72 = 3(5) + B$.

$B = -87$.

$$\therefore \frac{3x^2+23x-45}{(x+3)^2} = 3 + \frac{5}{x+3} - \frac{87}{(x+3)^2}.$$

9. (ii) $\therefore \int \frac{3x^2+25x-39}{(x+3)^2}\,dx = \int \frac{3x^2+23x-45+2x+6}{(x+3)^2}\,dx$

$= \int \left(\frac{3x^2+23x-45}{(x+3)^2} + \frac{2}{x+3}\right)dx$

$= \int 3 + \frac{5}{x+3} - \frac{87}{(x+3)^2} + \frac{2}{x+3}\,dx$

$= \int 3 + \frac{7}{x+3} - \frac{87}{(x+3)^2}\,dx$

$= 3x + 7\ln(x+3) + \frac{87}{x+3} + c$

for some constant, c.

10. (i) $3x-2y-6=0$.

$y=\frac{3}{2}x-3$.

Gradient of first diagonal $=\frac{3}{2}$.

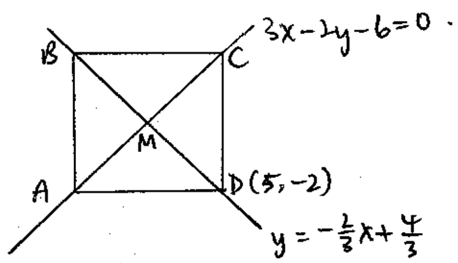
Gradient of second diagonal $=-\frac{2}{3}$ ($\perp$ diagonals of square).

Equation of second diagonal is

$$y-(-2)=-\frac{2}{3}(x-5)$$

$$y=-\frac{2}{3}x+\frac{4}{3}.$$

10. (ii)



Let D be (5, −2) and A, B and C be the other vertices.

$3x-2y-6=0$ ①

$y=-\frac{2}{3}x+\frac{4}{3}$ ②

① → ②: $3x-2\left(-\frac{2}{3}x+\frac{4}{3}\right)-6=0$.

$\frac{13}{3}x=\frac{26}{3} \Rightarrow x=2$

Subst. $x=2$ into ②: $y=-\frac{2}{3}(2)+\frac{4}{3}=0$.

Let M be centre of square, (2, 0).

Let B be (x_B, y_B).

Midpoint of BD is M.

$\Rightarrow \frac{x_B+5}{2}=2$, $\frac{y_B+(-2)}{2}=0$.

$x_B=-1$ $y_B=2$.

B is (−1, 2).

Length of MD $=\sqrt{(2-5)^2+(0+2)^2}$

$=\sqrt{13}=CM$ (diagonals of square bisect each other.)

Length of side CD of square $=\sqrt{MD^2+CM^2}$ (pythagoras' Theorem)

$=\sqrt{26}$

Let C be (x_C, y_C).

Since C falls on diagonal $3x-2y-6=0$.

$3x_C-2y_C-6=0$.

$y_C=\frac{3}{2}x_C-3$.

$\Rightarrow$ C is $(x_C, \frac{3}{2}x_C-3)$.

10. (ii) (continued)

$$CD = \sqrt{26}.$$

$$\sqrt{(x_c-5)^2+(\tfrac{3}{2}x_c-3+2)^2} = \sqrt{26}.$$

$$x_c^2 - 10x_c + 25 + \tfrac{9}{4}x_c^2 - 3x_c + 1 = 26.$$

$$\tfrac{13}{4}x_c^2 - 13x_c = 0.$$

$$\tfrac{13}{4}x_c(x_c-4) = 0.$$

$$x_c = 0 \quad \text{or} \quad x_c = 4.$$

$$y_c = \tfrac{3}{2}(0)-3 \quad \text{or} \quad y_c = \tfrac{3}{2}(4)-3$$
$$= -3 \qquad\qquad = 3.$$

Let A be (x_A, y_A)

Since $x_C > x_A$, $x_C = 4$ and $x_A = 0$.

$\Rightarrow y_c = 3$ and $y_A = -3$

$\therefore$ The other vertices of the square are $(0,-3)$, $(4,3)$ and $(-1,2)$.

11. (i) $\therefore f(x) = (\cos x - \sin x)(17\cos x - 7\sin x)$

$$= 17\cos^2 x - 7\sin x\cos x - 17\sin x\cos x + 7\sin^2 x$$

$$= 17\left(\frac{1+\cos 2x}{2}\right) - 24\sin x\cos x + 7\left(\frac{1-\cos 2x}{2}\right)$$

$$= \frac{17}{2} + \frac{17}{2}\cos 2x - 12\sin 2x + \frac{7}{2} - \frac{7}{2}\cos 2x$$

$$= 12 - 12\sin 2x + 5\cos 2x$$

$$= 12 + \sqrt{5^2+12^2}\cos\left(2x + \tan^{-1}\tfrac{12}{5}\right)$$

$$= 12 + 13\cos(2x+1.18), \quad R = 13, \; \alpha = 1.18 \text{ (3 s.f.)}$$

11. (ii)

$$-1 \leq \cos(2x+1.18) \leq 1.$$

$$-13 \leq 13\cos(2x+1.18) \leq 13$$

$$-1 \leq 12 + 13\cos(2x+1.18) \leq 25$$

$$-1 \leq f(x) \leq 25$$

$$13 \leq f(x)+14 \leq 39.$$

$$\frac{1}{39} \leq \frac{1}{f(x)+14} \leq \frac{1}{13}.$$

$$\frac{78}{39} \leq \frac{78}{f(x)+14} \leq \frac{78}{13}$$

$$2 \leq \frac{78}{f(x)+14} \leq 6.$$

$\therefore$ Greatest value of $\frac{78}{f(x)+14} = 6$, least value of $\frac{78}{f(x)+14} = 2$.

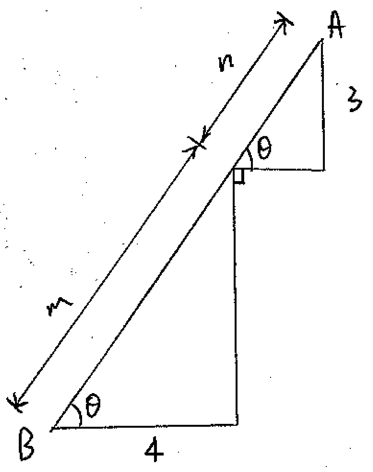
11. (iii) Greatest value of $\frac{78}{f(x)+14} = 6$ when $\cos(2x+1.18) = -1$.

$$2x + 1.18 = \cos^{-1}(-1) = \pi$$

$$\therefore x = \frac{\pi - 1.18}{2}$$

$$= 0.983 \text{ (3 s.f.)}$$

12. (i)



$$\frac{3}{n} = \sin\theta. \quad \text{and} \quad \frac{4}{m} = \cos\theta$$

$$n = \frac{3}{\sin\theta} \qquad m = \frac{4}{\cos\theta}$$

$\therefore$ length of AB, $L = m + n$

$$= \frac{4}{\cos\theta} + \frac{3}{\sin\theta} \quad \text{. (shown).}$$

12. (ii) $\therefore \frac{dL}{d\theta} = -\frac{4}{\cos^2\theta}(-\sin\theta) - \frac{3}{\sin^2\theta}(\cos\theta)$

$$= \frac{4\sin\theta}{\cos^2\theta} - \frac{3\cos\theta}{\sin^2\theta}$$

12. (iii) When $\frac{dL}{d\theta} = 0$,

$$\frac{4\sin\theta}{\cos^2\theta} = \frac{3\cos\theta}{\sin^2\theta}$$

$$4\sin^3\theta = 3\cos^3\theta.$$

$$\tan^3\theta = \frac{3}{4}.$$

$$\tan\theta = \sqrt[3]{\frac{3}{4}}$$

$$\therefore \theta = \tan^{-1}\sqrt[3]{\frac{3}{4}} = 42.3° \text{ (1 d.p.)}$$

12. (iv)

$\theta°$	42.3^-	42.3	42.3^+
sign of $\frac{dL}{d\theta}$	\	—	/

(minimum)

When $\theta = 42.3°$ (1 d.p.), $L = 9.87$m (minimum length)

$\therefore$ Advice is to ensure metal tubes transported are no longer than 9.87m in order to move around from a corridor to another.