Montfort Prelim 4 AM 2017 P2

1. $\log_2 \sqrt{5x+1} - \log_4 64 + 2\log_9 3 = \log_4 (x-2)$

$\frac{\log_4 \sqrt{5x+1}}{\log_4 2} - 3 + 2\left(\frac{1}{2}\right) = \log_4 (x-2)$

$2\log_4 (5x+1)^{\frac{1}{2}} - \log_4 (x-2) = 2.$

$\log_4 (5x+1) - \log_4 (x-2) = 2.$

$\log_4 \frac{5x+1}{x-2} = 2.$

$\frac{5x+1}{x-2} = 4^2$

$5x+1 = 16x - 32$

$11x = 33$

$\therefore x = 3$

2. (i) $ax^2 + bx + 6 = 0,$

$\frac{1}{\alpha} + \frac{1}{\beta} = -\frac{b}{a}.$ ①

$\left(\frac{1}{\alpha}\right)\left(\frac{1}{\beta}\right) = \frac{6}{a}$ ②

①: $\frac{\alpha+\beta}{\alpha\beta} = -\frac{b}{a}$

Subst. $\alpha+\beta = \frac{1}{2}$, $\alpha\beta = 4$,

$\frac{\left(\frac{1}{2}\right)}{4} = -\frac{b}{a}.$

$a = -8b.$ ③

Subst ③ and $\alpha\beta = 4$ into ②: $\frac{1}{\alpha\beta} = \frac{6}{(-8b)}$

$\Rightarrow b = -3$

$\Rightarrow a = -8(-3)$

$= 24$

$\therefore a = 24, b = -3.$

2. (ii) $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$

$= (\frac{1}{2})^2 - 2(4)$

$= -\frac{31}{4}$

$\alpha^3+\beta^3 = (\alpha+\beta)(\alpha^2-\alpha\beta+\beta^2)$

$= (\frac{1}{2})(-\frac{31}{4}-4)$

$= -\frac{47}{8}.$

$\alpha^3\beta^3 = (\alpha\beta)^3$

$= (4)^3$

$= 64.$

$\therefore$ Equation with roots α^3 and β^3 is

$x^2 - (-\frac{47}{8})x + 64 = 0.$

$8x^2+47x+512=0.$ where $m=8, n=47, p=512.$

3. (i) $y = 2(5-x)(3x+1)^{-1}$, $x \neq -\frac{1}{3}$

$\frac{dy}{dx} = 2\left[(-1)(3x+1)^{-1} - (5-x)(3x+1)^{-2}(3)\right]$

$= 2(3x+1)^{-2}(-3x-1-15+3x)$

$= -\frac{32}{(3x+1)^2}.$

Since $(3x+1)^2>0$, $-\frac{32}{(3x+1)^2}<0.$

$\Rightarrow \frac{dy}{dx}<0$, $\therefore$ y is a decreasing function, $x\neq-\frac{1}{3}.$

3. (ii) Given $\frac{dy}{dt} = -8\frac{dx}{dt}$,

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

$= -8.$

$\Rightarrow -\frac{32}{(3x+1)^2} = -8.$

$3x+1 = 2$ or $-2.$

$x = \frac{1}{3}$ or $-1.$

$\therefore y = \frac{14}{3}$ or $-6.$

4. (i) $u = 3^x$.

$\therefore 7 - 3^{2x+1} - 2(3^{-x} + 3^x)$

$= 7 - (3^x)^2 \times 3 - 2\left(\frac{1}{3^x} + 3^x\right)$

$= 7 - 3u^2 - 2\left(\frac{1}{u} + u\right)$

$= 7 - 3u^2 - \frac{2}{u} - 2u$.

4. (ii) $7 - 3^{2x+1} - 2(3^{-x} + 3^x) = 0$.

$\Rightarrow 7 - 3u^2 - \frac{2}{u} - 2u = 0$ where $u = 3^x$.

$3u^2 + 2u - 7 + \frac{2}{u} = 0$.

$3u^3 + 2u^2 - 7u + 2 = 0$.

Let $f(u) = 3u^3 + 2u^2 - 7u + 2$

When $u = 1$, $f(1) = 3(1)^3 + 2(1)^2 - 7(1) + 2 = 0$

$\Rightarrow$ $u - 1$ is a factor of $f(u)$

Let $f(u) = 3u^3 + 2u^2 - 7u + 2$

$= (u-1)(3u^2 + au - 2)$ for some constant, a

Comparing terms in u,

$-7u = -2u - au$

$a = 5$

$f(u) = 3u^3 + 2u^2 - 7u + 2$

$= (u-1)(3u^2 + 5u - 2)$

$= (u-1)(3u-1)(u+2)$

When $f(u) = 0$,

$(u-1)(3u-1)(u+2) = 0$.

$u = 1$, $u = \frac{1}{3}$ or $u = -2$

$\Rightarrow 3^x = 1$, $3^x = \frac{1}{3}$ or $3^x = -2$ (rej. $\because 3^x > 0$)

$\therefore x = 0$ or $x = -1$.

5. (i) General term in $\left(-2x^2 - \frac{1}{2x}\right)^9$ is

$$\binom{9}{r}(-2x^2)^{9-r}\left(-\frac{1}{2x}\right)^r$$
$$= \binom{9}{r}(-2)^{9-r}(x)^{18-2r}\left(-\frac{1}{2}\right)^r(x)^{-r}$$
$$= \binom{9}{r}(-2)^{9-r}\left(-\frac{1}{2}\right)^r(x)^{18-3r}$$

For term independent of x, $18-3r=0$.

$\Rightarrow r=6$.

$\therefore$ Term independent of x

$$= \binom{9}{6}(-2)^{9-6}\left(-\frac{1}{2}\right)^6$$
$$= -\frac{21}{2}.$$

5. (ii) Let $18-3r=-3$.

$\Rightarrow r=7$.

$$\text{Term in } \frac{1}{x^3} = \binom{9}{7}(-2)^{9-7}\left(-\frac{1}{2}\right)^7 \cdot \frac{1}{x^3}$$
$$= -\frac{9}{8x^3}$$

$$\left(\frac{5}{4} - 4x^3\right)\left(-2x^2 - \frac{1}{2x}\right)^9$$
$$= \left(\frac{5}{4} - 4x^3\right)\left(-\frac{21}{2} - \frac{9}{8x^3} + \dots\right)$$
$$= \left(\frac{5}{4}\right)\left(-\frac{21}{2}\right) + (-4x^3)\left(-\frac{9}{8x^3}\right) + \dots$$
$$= -\frac{69}{8} + \dots$$

$\therefore$ Term independent of $x = -\frac{69}{8}$.

6. (i) $B\hat{E}F = E\hat{C}F$ (alternate segment theorem)

$B\hat{F}E = E\hat{F}C$ (common angle)

$\therefore$ By AA property, ΔBEF is similar to ΔECF (proven).

6. (ii) $B\hat{F}E = B\hat{C}E$ ($EF = EC$, base $\angle$s of isosceles ΔCEF.)

$= B\hat{A}D$ ($\angle$s in the same segment)

$C\hat{E}G = \angle B\hat{F}E + B\hat{C}E$ (ext. $\angle$s of ΔCEF.)

$= 2B\hat{C}E$

$= 2B\hat{A}D$

$C\hat{E}G = C\hat{A}D$ (alternate segment theorem)

$\therefore C\hat{A}D = 2B\hat{A}D$ (proven)

6. (iii) Since $\triangle BEF$ and $\triangle ECF$ are similar,

$$\frac{FE}{FC} = \frac{FB}{FE}$$

$\therefore FE^2 = FB \times FC$ (proven).

Given $FE = CE$,

$\therefore CE \times FE = FB \times FC$ (proven)

7. (i) $(\cos 2x + \cos x)^2 + (\sin 2x + \sin x)^2$

$= \cos^2 2x + 2\cos x \cos 2x + \cos^2 x + \sin^2 2x + 2\sin x \sin 2x + \sin^2 x$

$= 1 + 1 + 2\cos x(2\cos^2 x - 1) + 2\sin x(2\sin x \cos x)$

$= 2 + 4\cos^3 x - 2\cos x + 4\sin^2 x \cos x$

$= 2 + 4\cos^3 x - 2\cos x + 4(1 - \cos^2 x)\cos x$

$= 2 + 4\cos^3 x - 2\cos x + 4\cos x - 4\cos^3 x$

$= 2 + 2\cos x$

$\therefore A = 2,\ B = 2.$

7. (ii) $(\cos 2x + \cos x)^2 + (\sin 2x + \sin x)^2 + 1 = \dfrac{2}{\operatorname{cosec}^2 x} - \cos x$

$2 + 2\cos x + 1 = 2\sin^2 x - \cos x$

$3 + 3\cos x = 2(1 - \cos^2 x)$

$2\cos^2 x + 3\cos x + 1 = 0.$

$(2\cos x + 1)(\cos x + 1) = 0.$

$\cos x = -\frac{1}{2}$ or $\cos x = -1$

basic $\angle$ $x = \cos^{-1}(\frac{1}{2})$ $\quad x = -180°, 180°.$

$= 60°.$

$x = -180° + 60°,\ 180° - 60°$

$= -120°,\ 120°.$

$\therefore x = -180°, -120°, 120°, 180°.$

8.(i) $P = P_0 e^{-kt}$

$\ln P = \ln P_0 + \ln e^{-kt}$

$\ln P = \ln P_0 - kt$

Plot $\ln P$ against t:

gradient of line $= -k$

intercept on $\ln P$-axis $= \ln P_0$.

t	10	20	30	40	50
P	245	155	88	55	33
$\ln P$	5.50	5.04	4.48	4.01	3.50

gradient of straight line $= -\dfrac{5.50 - 4.50}{30 - 10}$

$= -0.05$.

$\Rightarrow -k = -0.05$

$k = 0.05$.

intercept on $\ln P$-axis $= 6.00$

$\ln P_0 = 6.00$

$P_0 = e^{6.00}$

$= 403.$ (3s.f.)

$\therefore P_0 = 403, \quad k = 0.05$.

8.(ii) when $t = 0$, $P = P_0$

when $P = \frac{1}{5}P_0$,

$\ln P = \ln \frac{1}{5} P_0$

$= \ln\frac{1}{5} + \ln P_0$

$= -1.609 + 6.00$

$= 4.391$

when $\ln P = 4.391 \approx 4.40$, $t = 32$.

$\therefore$ Estimated time taken $= 32$ min.

8.(iii) P approaches zero since $e^{-kt} = \dfrac{1}{e^{kt}}$,

and as t becomes large, e^{kt} becomes large,

$\dfrac{1}{e^{kt}}$ approaches zero.

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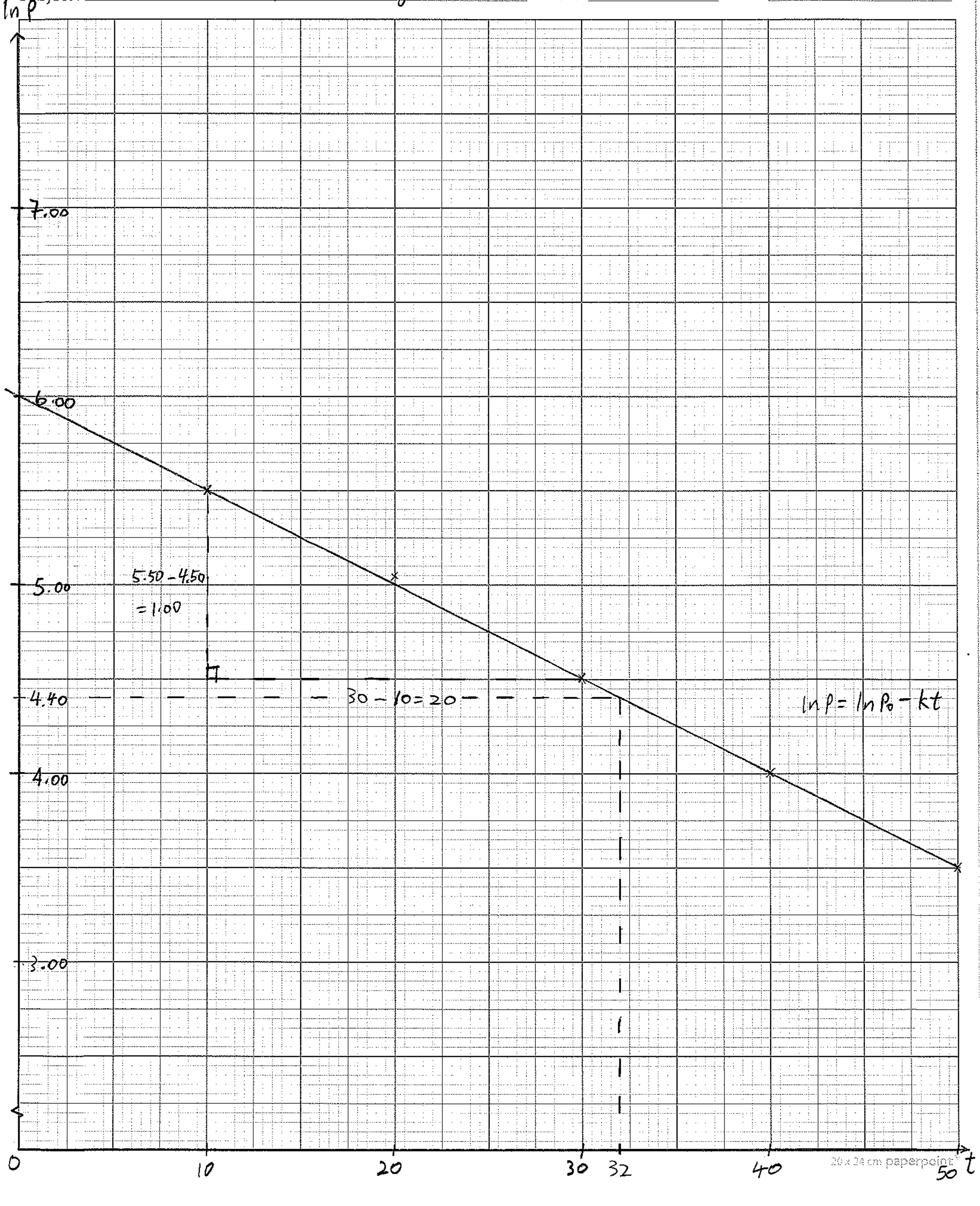


9. (i) $C_1: x^2+y^2-18x-4y-15=0.$

$$x^2-18x+y^2-4y=15$$

$$x^2-18x+9^2+y^2-4y+2^2=15+9^2+2^2$$

$$(x-9)^2+(y-2)^2=100=10^2.$$

$\therefore$ centre of C_1 is $A(9,2)$.

$\therefore$ radius of C_1 is 10 units.

9. (ii) Gradient of straight line $= -\tan\theta$

$$= -\frac{3}{4}$$

Equation of straight line is

$$y-(-6) = -\frac{3}{4}(x-3)$$

$$y = -\frac{3}{4}x-\frac{15}{4} \quad ①$$

$$x^2+y^2-18x-4y-15=0 \quad ②$$

① → ②: $x^2+\left(-\frac{3}{4}x-\frac{15}{4}\right)^2-18x-4\left(-\frac{3}{4}x-\frac{15}{4}\right)-15=0.$

$$x^2+\frac{9}{16}x^2+\frac{45}{8}x+\frac{225}{16}-18x+3x+15-15=0.$$

$$\frac{25}{16}x^2-\frac{75}{8}x+\frac{225}{16}=0.$$

$$x^2-6x+9=0.$$

Discriminant $= (-6)^2-4(1)(9)$

$= 0.$

$\therefore$ Straight line $y=-\frac{3}{4}x-\frac{15}{4}$ is a tangent to C_1.

9. (iii) $\therefore$ Equation of tangent to the circle at D is $y=12$.

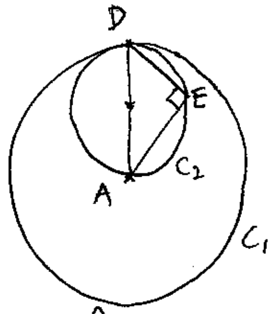
9. (iv) Subst. $y=12$ into $y=-\frac{3}{4}x-\frac{15}{4}$

$$-\frac{3}{4}x-\frac{15}{4}=12.$$

$$x=-21$$

$\therefore$ Point of intersection between tangents at B and D is $(-21,12)$.

9. (v)



Since $D\hat{E}A = 90°$, AD is a diameter of C_2 (right $\angle$ in semicircle).

Midpoint of AD is $\left(\frac{9+9}{2}, \frac{2+12}{2}\right) = (9, 7)$ (Centre of C_2).

$AD = 10$ units (radius of C_1)

Radius of $C_2 = \frac{10}{2} = 5$ units.

$\therefore$ Equation of C_2 is

$(x-9)^2 + (y-7)^2 = 25$.

10. (i) $a = -5e^{-t}$

$v = \int a \, dt$

$= \int -5e^{-t} \, dt$

$= 5e^{-t} + c$ for some constant, c

At $t = 0$, $v = 4.8$

$5e^0 + c = 4.8$

$c = -0.2$

$\Rightarrow v = 5e^{-t} - 0.2$

At P, $v = 0$.

$5e^{-t} - 0.2 = 0$.

$e^{-t} = 0.04$

$-t = \ln 0.04$

$\therefore t = -\ln 0.04$ (shown).

10. (ii) Distance travelled by particle from O to P

$= \int_0^{-\ln 0.04} v \, dt$

$= \left[-5e^{-t} - 0.2t\right]_0^{-\ln 0.04}$

$= 4.16$ m (3 s.f.)

$\therefore$ Distance of particle from O at P = 4.16 m.

11. (i)(a) $y = (3x^2 - 3x + 1)(x-1)$.

$= 3x^3 - 3x^2 + x - 3x^2 + 3x - 1$

$= 3x^3 - 6x^2 + 4x - 1$

$\frac{dy}{dx} = 9x^2 - 12x + 4$

$= (3x-2)^2$.

Since $(3x-2)^2 \geqslant 0$,

$\frac{dy}{dx} \geqslant 0$.

$\therefore$ tangents to the curve can never have negative gradient.

11. (i)(b) when $x = \frac{2}{3}$, $\frac{dy}{dx} = \left[3(\frac{2}{3}) - 2\right]^2$

$= 0$.

Gradient of tangent to curve at $x = \frac{2}{3}$ is 0

$\Rightarrow \therefore$ tangent at $x = \frac{2}{3}$ is parallel to x-axis.

* 11. (i)(c)

x	$\frac{2}{3}^-$	$\frac{2}{3}$	$\frac{2}{3}^+$
Sign of $\frac{dy}{dx}$	/	—	/

$\therefore$ Nature of the point of the curve at $x = \frac{2}{3}$ is an inflexion point.

11. (i)(d) When $x = 2$, $\frac{dy}{dx} = 16$, $y = 7$.

Gradient of normal at $x = 2$ is $-\frac{1}{16}$.

Equation of normal at $x = 2$ is

$y - 7 = -\frac{1}{16}(x-2)$.

$y = -\frac{1}{16}x + 7\frac{1}{8}$.

Since the y intercept of the normal is $7\frac{1}{8}$,

$\therefore$ normal to the curve at $x = 2$ cuts the y axis at $7\frac{1}{8}$.

11. (ii) $y = \frac{1}{20}(x+2)^2(x-5)$

$= \frac{1}{20}(x^2+4x+4)(x-5)$

$= \frac{1}{20}(x^3 - x^2 - 16x - 20)$

$= \frac{1}{20}x^3 - \frac{1}{20}x^2 - \frac{4}{5}x - 1.$

$\therefore$ Area of shaded regions

$= \int_0^5 \frac{1}{5}x - 1 - \left(\frac{1}{20}x^3 - \frac{1}{20}x^2 - \frac{4}{5}x - 1\right) dx + \int_5^{6.5} \frac{1}{20}x^3 - \frac{1}{20}x^2 - \frac{4}{5}x - 1 - \left(\frac{1}{5}x - 1\right) dx$

$= \int_0^5 -\frac{1}{20}x^3 + \frac{1}{20}x^2 + x \, dx + \int_5^{6.5} \frac{1}{20}x^3 - \frac{1}{20}x^2 - x \, dx$

$= \left[-\frac{1}{80}x^4 + \frac{1}{60}x^3 + \frac{1}{2}x^2\right]_0^5 + \left[\frac{1}{80}x^4 - \frac{1}{60}x^3 - \frac{1}{2}x^2\right]_5^{6.5}$

$= \frac{325}{48} - 0 + \left[-\frac{13013}{3840} - \left(-\frac{325}{48}\right)\right]$

$= 10\frac{587}{3840}$ or 10.2 units^2 (3 s.f.)